MTH5129 Probability & Statistics II

Coursework 3

1. Suppose X and Y have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } x > y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the conditional density function $f_{Y|X=x}(y)$.
In lectures we calculated the $f_{X|Y=y}$.

Solution: By definition

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

whenever $f_X(x) > 0$. Hence we need to find the marginal density f_X. If $x \leq 0$ then $f_x(x) = 0$. If $x > 0$ we have

$$\begin{aligned} f_x(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,t)\,dt \\ &= \int_{-\infty}^{0} f_{X,Y}(x,t)\,dt + \int_{0}^{x} f_{X,Y}(x,t)\,dt + \int_{x}^{\infty} f_{X,Y}(x,t)\,dt \\ &= \int_{0}^{x} 2e^{-x-t}\,dt \\ &= \left[-2e^{-x-t}\right]_{t=0}^{x} \\ &= 2e^{-x} - 2e^{-2x}. \end{aligned}$$

Hence, if $x \leq 0$ then $f_{Y|X=x}$ is not defined. If $x > 0$ then

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

If $y > x$ or $y < 0$ then $f_{X,Y}(x,y) = 0$ and so $f_{Y|X=x}(y) = 0$.

Finally (the interesting case!) if $0 < y < x$:

$$f_{Y|X=x}(y) = \frac{2e^{-x-y}}{2e^{-x} - 2e^{-2x}} = \frac{e^{-y}}{1 - e^{-x}}.$$

Hence, putting it all together: $f_{Y|X=x}(y)$ is defined only for $x > 0$ and it is given by

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{e^{-y}}{1 - e^{-x}} & \text{if } x > y > 0 \\ 0 & \text{otherwise} \end{cases}$$

2. Suppose that X, Y, Z are random variables. If $a \in \mathbb{R}$, prove that

   a) $\mathrm{Cov(aX, Y) = Cov(X, aY) = a\, Cov(X, Y)}$
   b) $\mathrm{Cov(a + X, Y) = Cov(X, a + Y) = Cov(X, Y)}$
   c) $\mathrm{Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)}$

   Solution: All parts can be proved by using the definition of Covariance (see definition and procedure followed in lecture).

   Below we prove part (a) – other equations follow similarly:

$$
\begin{aligned}
\mathrm{Cov}(aX, Y) &= E\left[(aX - E(aX))(Y - E(Y))\right] \\
&= E\left[(aX - aE(X))(Y - E(Y))\right] \\
&= E\left[a(X - E(X))(Y - E(Y))\right] \\
&= aE\left[(X - E(X))(Y - E(Y))\right] \\
&= a\mathrm{Cov}(X, Y)
\end{aligned}
$$

3. Suppose we throw a dice. Define the events A to be "the outcome is an odd number", B to be "the outcome is 2", C to be "the outcome is either a 5 or a 6".

   a) What is the sample space Ω?
   b) Are the events A and B independent?
   c) Are the events A and C independent?
   d) Are the events A,B and C independent?

   Solution:

   a) This question translates to "What are the possible outcomes from throwing a dice?" The answer is

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

   b) We have

$$P(A) = 3/6 = 1/2$$

   and

$$P(B) = 1/6$$

   while

$$P(A \cap B) = 0$$

   which means that

$$P(A \cap B) \neq P(A) \cdot P(B)$$

   hence the events A and B are not independent.

c) We have

$$P(C) = 2/6 = 1/3$$

and

$$P(A \cap C) = P(Outcome = 5) = 1/6$$

which means that

$$P(A \cap C) = P(A) \cdot P(C)$$

hence the events A and C are independent.

d) Since we know that the events A and B are not independent, we immediately have that the events A,B and C are not independent.

4. Suppose that X, Y are discrete, independent random variables. You are reminded that this means that for any x, y from the range of X and Y

$$P(X = x,\ Y = y) = P(X = x) \times P(\ Y = y).$$

Find $E(XY \mid Y = y)$.

Solution: To calculate $E(XY|\{Y = y\})$ note that

$$\begin{aligned}
E(XY|\{Y = y\}) &= E(Xy|\{Y = y\}) = yE(X|\{Y = y\}) \\
&\qquad\qquad \text{(by property of expectations)} \\
&= y\sum_x xP(X = x \mid \{Y = y\}) \\
&\qquad\qquad \text{(by the definition of } E(X|\{Y = y\})) \\
&= y\sum_x xP(X = x) \qquad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= yE(X) \qquad \text{(by the definition of } E(X)).
\end{aligned}$$

5. Suppose X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} e^{-x-y} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find whether X and Y are independent.

Solution: We present two ways of solving this question.

One way: Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ and show that $f_X(x) \cdot f_Y(y) = f_{X,Y}(x, y)$.

Another way: Let

$$g(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(y) = \begin{cases} e^{-y} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

What is $g(x)h(y)$?

If $x \leq 0$ or $y \leq 0$ then $g(x)h(y) = 0$ which equals $f_{X,Y}(x, y)$ in this case.

If $x > 0$ and $y > 0$ then

$$g(x)h(y) = e^{-x}e^{-y} = e^{-x-y} = f_{X,Y}(x, y).$$

In all cases $g(x)h(y) = f_{X,Y}(x, y)$ so X and Y are independent.