Queen Mary
University of London

Main Examination period 2018

MTH6128 / MTH6128P: Number Theory

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: S. Lester, X. Li

Question 1. [20 marks]

(a) Define the terms **algebraic integer**, **quadratic integer**, and **transcendental**. **[6]**

(b) Determine which of the following are quadratic integers. Explain which theorems you have used. **[8]**

(i) $\frac{1+\sqrt{49}}{2}$;

(ii) $\frac{\sqrt{3}}{2} - \frac{7}{2}$;

(iii) $\frac{\sqrt{5}}{2} + \frac{\sqrt{-3}}{2}$;

(iv) $\frac{7}{2} + \frac{\sqrt{65}}{2}$.

(c) Let D be a natural number which is not a square. Using minimal polynomials, show that $\frac{1+\sqrt{D}}{2}$ is an algebraic integer if and only if $D \equiv 1 \pmod 4$. **[6]**

Question 2. [20 marks]

(a) What is a **periodic continued fraction**? Give an example of an irrational number whose continued fraction expansion is not periodic. **[4]**

(b) Use the Euclidean algorithm to find a continued fraction expansion for $\frac{241}{113}$. **[5]**

(c) Determine the value of the infinite continued fraction

$$[1; \overline{2, 1}].$$ **[5]**

Write your answer in the form $u + v\sqrt{d}$, where $u, v \in \mathbb{Q}$ and $d \in \mathbb{Z}$.

(d) Find the continued fraction expansion of $\sqrt{7}$. **[6]**

Question 3. [20 marks]

(a) Given that

$$\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$$

find the fundamental solution to the equation

$$x^2 - 29y^2 = \pm 1.$$

Use your answer to write down all positive integer solutions to the equation $x^2 - 29y^2 = \pm 1$. Explain why you have found ALL solutions.

[8]

(b) Given that $37^2 \equiv -1 \pmod{137}$ use Hermite's algorithm to find integers x, y such that

$$x^2 + y^2 = 137.$$

[8]

(c) Suppose that $n \equiv 3 \pmod 4$. Show that $x^2 + y^2 = n$ has no integer solutions. **[4]**

Question 4. [20 marks]

(a) Given a positive integer n define the **order of** $x \pmod{n}$. Define the term **primitive root** $\pmod{p}$. **[4]**

(b) Find a primitive root $\pmod{13}$. How many primitive roots $\pmod{13}$ are there? **[4]**

(c) Does there exist an integer n such that $n^4 \not\equiv 1 \pmod{17}$ and $n^5 \equiv 1 \pmod{17}$? Justify your answer by stating explicitly which theorems you use in the proof. **[6]**

(d) Compute $\varphi(280)$. (Hint: $280 = 2^3 \cdot 5 \cdot 7$.) **[3]**

(e) Show that $\varphi(n)$ is even for $n > 2$. **[3]**

Question 5. [20 marks]

(a) Define the term **quadratic residue**. Define the **Legendre symbol** $\left(\frac{a}{p}\right)$. State the **Law of Quadratic Reciprocity**. **[6]**

(b) Both 227 and 137 are primes. Compute $\left(\frac{137}{227}\right)$. You should clearly state any rules you use for calculating the Legendre symbol. **[7]**

(c) Let p be an odd prime. Suppose that $p + 2$ is also prime. Show that p is a quadratic residue $\pmod{(p+2)}$ if and only if

$$p \equiv \pm 1 \pmod{8}.$$

[7]

End of Paper.