Alternative assessment 2020 solutions

(1) $\underline{z} = -3\underline{x} - 2\underline{y}$, $\underline{w} = -6\underline{x} - 3\underline{y} - 2\underline{z} = -6\underline{x} - 3\underline{y} - 2(-3\underline{x} - 2\underline{y})$

$$= \cancel{-6x} - 3y \cancel{+6x} + 4y = \underline{y}$$

$$\Downarrow$$

$\underline{z} \in \mathrm{Span}(\underline{x}, \underline{y})$

$\underline{w} = \underline{y}$ so $\mathrm{Span}(\underline{w}) = \mathrm{Span}(\underline{y})$

$$\mathrm{Span}(\underline{x}, \underline{z}) \overset{?}{=} \mathrm{Span}(\underline{y}, \underline{w}) = \mathrm{Span}(\underline{y})$$

" $\mathrm{Span}(\underline{x}, \underline{y})$

$\underline{y} \in \mathrm{Span}(\underline{x}, \underline{z})$

Question reduces to asking: is

$$\mathrm{Span}(\underline{x}, \underline{y}) = \mathrm{Span}(\underline{y}) \ ?$$

This would only be true if $\underline{x}, \underline{y}$ lin. dep.

i.e. if $\underline{x}$ is a scalar mult. of $\underline{y}$

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if $\underbrace{\underline{x}, \underline{y} \text{ are lin. indep.}}$

then $\mathrm{Span}(\underline{x}, \underline{y}) \supsetneq \mathrm{Span}(\underline{y})$

if $\underbrace{\underline{x}, \underline{y} \text{ are lin. dep.}}$

$\Rightarrow \mathrm{Span}(\underline{x}, \underline{y}) = \mathrm{Span}(\underline{y})$

or $= \mathrm{Span}(\underline{x})$

② (1) Note:
$$\underline{\begin{pmatrix} 16 & 8 \\ 24 & 8 \end{pmatrix}} = (-4)\cdot\underline{\begin{pmatrix} -4 & -2 \\ -6 & -2 \end{pmatrix}}$$

so NOT lin. indep.

(2). By Steinitz theorem, max. no. of lin. indep. vectors in $\mathbb{R}^{2\times2}$ is $\dim \mathbb{R}^{2\times 2} = 4$. We have 5 matrices, so they must be $\underline{\text{lin. dep.}}$

(3) Supp. $\alpha\begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix} + \beta\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix} + \gamma\begin{pmatrix} -4 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{aligned} \alpha - 4\beta - 4\gamma &= 0 \\ -4\alpha + \beta - 2\gamma &= 0 \\ -2\alpha + \beta + \gamma &= 0 \\ 3\alpha - 2\beta + 0\cdot\gamma &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -4 & -4 & 0 \\ -4 & 1 & -2 & 0 \\ -2 & 1 & 1 & 0 \\ 3 & -2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

All variables LEADING

$\Rightarrow \alpha = 0,\ \beta = 0,\ \gamma = 0$

so LIN. INDEP.

(4) Note $\begin{pmatrix} 16 & 32 \\ 16 & 8 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} -4 & -2 \\ -6 & -2 \end{pmatrix}$

hence lin. indep.

③ lin. indep. test:

$$\begin{pmatrix} -4 & 1 & 4 \\ -2 & -1 & 4 \\ 5 & 2 & -2 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

no free variables, all vars. leading

$\rightsquigarrow$ LIN. INDEP.

(4) Supp. $\underline{u}_4 \notin \text{Span}(\underline{u}_1, \underline{u}_2, \underline{u}_3)$

(A) $\overset{?}{\Rightarrow} \{\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4\}$ lin. dependent ?

NO, if $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ lin. indep $\Rightarrow \{u_1, u_2, u_3, u_4\}$ lin. indep.

(B) if none of u_1, u_2, u_3 is $\underline{0}$, then $\{u_1, u_2, u_3, u_4\}$ lin. indep.?

NO, becaus $u_1\ u_2\ u_3$ could be non. zero, but still lin. dep.

(C) $\{u_1, u_2, u_3, u_4\}$ could be lin. dep or lin. indep depending on the vector space chosen

NO, the claim does not make sense.

(D) $\{u_1, u_2, u_3, u_4\}$ is never lin. dep.

NO, u_1, u_2, u_3 could be dependent.

(E) $\{u_1, u_2, u_3, u_4\}$ is lin. dep $\Leftrightarrow \{u_1, u_2, u_3\}$ lin. dep.

TRUE: easier to understand the equivalent statement

$\{u_1\ u_2\ u_3\ u_4\}$ is lin. indep $\Leftrightarrow \{u_1, u_2, u_3\}$ lin. indep.

(F) none

Follows from some proofs in the lecture

Try proving directly

$\boxed{\Rightarrow}$ clear.

$\boxed{\Leftarrow}$ $\{u_1, u_2, u_3\}$ lin. indep & $u_4 \notin \text{Span}(u_1, u_2, u_3)$

$\Rightarrow \{u_1, u_2, u_3, u_4\}$ is lin. indep.

(5) Find a basis for the space H of 2×2 lower-triang. matrices.

Principle: try to write a "generic" element of the space.

In this case, the most general 2×2 lower-triang. matrix is of

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So H is spanned by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

observe that are lin. indep

$\Rightarrow$ form a basis for H

(6) $L : \mathbb{R}^3 \to \mathbb{R}^3$ linear map s.t.

$$L\Big(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{=\,\underline{e}_1}\Big) = \begin{bmatrix}4\\3\\-4\end{bmatrix}, \quad L\Big(\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{=\,\underline{e}_2}\Big) = \begin{bmatrix}3\\-1\\-1\end{bmatrix}, \quad L\Big(\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{=\,\underline{e}_3}\Big) = \begin{bmatrix}1\\-4\\1\end{bmatrix}$$

Then $L\left(\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right) = ?$ $\quad \begin{bmatrix}-5\\-10\\7\end{bmatrix}$

<u>1st solution</u>: using the <u>defn. of linear map.</u>

$$L\left(\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right) = L\left(-1\cdot\underline{e}_1 - 1\cdot\underline{e}_2 + 2\cdot\underline{e}_3\right) \overset{\text{linear}}{=} -L(\underline{e}_1) - L(\underline{e}_2) + 2L(\underline{e}_3)$$

$$= -\begin{bmatrix}4\\3\\-4\end{bmatrix} - \begin{bmatrix}3\\-1\\-1\end{bmatrix} + 2\begin{bmatrix}1\\-4\\1\end{bmatrix} = \begin{bmatrix}-5\\-10\\7\end{bmatrix}$$

<u>2nd solution</u>: using "associated matrix"

$E = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ std. basis for $\mathbb{R}^3$

$[L]_E$ has column $L(\underline{e}_1), L(\underline{e}_2), L(\underline{e}_3)$

$$= \begin{pmatrix}4 & 3 & 1\\3 & -1 & -4\\-4 & -1 & 1\end{pmatrix}$$

$$L\left(\begin{bmatrix}-1\\-1\\2\end{bmatrix}\right) = [L]_E \cdot \begin{bmatrix}-1\\-1\\2\end{bmatrix} = \begin{pmatrix}4 & 3 & 1\\3 & -1 & -4\\-4 & -1 & 1\end{pmatrix}\begin{pmatrix}-1\\-1\\2\end{pmatrix} = \begin{bmatrix}-5\\-10\\7\end{bmatrix}$$

⑦ $L : \mathbb{R}^2 \to \mathbb{R}^2$ lin. map s.t.

$$L\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \quad L\left(\begin{bmatrix} -6 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 33 \\ -3 \end{bmatrix}.$$

Find matrix $[L]_E$ where $E = \left(\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ std. basis.

Did not fully cover the material this year, but we can still solve it:

$$[L]_E = \Big(\underbrace{[L(\underline{e}_1)]_E}_{\underline{u}}, \underbrace{[L(\underline{e}_2)]_E}_{\underline{v}} \Big)$$; need to calculate $\underline{u}, \underline{v}$

Know:

$$\begin{bmatrix} -3 \\ 9 \end{bmatrix} = L\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = L(2\underline{e}_1 - \underline{e}_2) = 2L(\underline{e}_1) - L(\underline{e}_2) = 2\underline{u} - \underline{v}$$

$$\begin{bmatrix} 33 \\ -3 \end{bmatrix} = L\left(\begin{bmatrix} -6 \\ -5 \end{bmatrix}\right) = L(-6\underline{e}_1 - 5\underline{e}_2) = -6L(\underline{e}_1) - 5L(\underline{e}_2) = -6\underline{u} - 5\underline{v}$$

$$\rightsquigarrow \quad \begin{aligned} 2\underline{u} - \underline{v} &= \begin{bmatrix} -3 \\ 9 \end{bmatrix} \\ -6\underline{u} - 5\underline{v} &= \begin{bmatrix} 33 \\ -3 \end{bmatrix} \end{aligned} \qquad \left(\begin{aligned} 2u_1 - v_1 &= -3 \\ 2u_2 - v_2 &= 9 \\ -6u_1 - 5v_1 &= 33 \\ -6u_2 - 5v_2 &= -3 \end{aligned} \right) \Rightarrow \quad \underline{u} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$\Rightarrow [L]_E = \begin{pmatrix} -3 & -3 \\ 3 & -3 \end{pmatrix}$$

To check that this is correct, verify:

$$\begin{pmatrix} -3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \cdots = \begin{pmatrix} -3 \\ 9 \end{pmatrix}, \quad \begin{pmatrix} -3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} -6 \\ -5 \end{pmatrix} = \begin{pmatrix} 33 \\ -3 \end{pmatrix}$$

⑧ $E = \left(A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

standard basis for $V = \mathbb{R}^{2\times 2}$.

Let $L : V \to V$, $L(A) = 9A + 2A^T$ lin. map.

Find $[L]_E = ?$

Know: columns are $[L(A_1)]_E, [L(A_2)]_E, [L(A_3)]_E, [L(A_4)]_E$

$$L(A_1) = 9A_1 + 2A_1^T = 9A_1 + 2A_1 = 11A_1$$

$$= 11A_1 + 0A_2 + 0A_3 + 0A_4 \rightsquigarrow [L(A_1)]_E = \begin{pmatrix} 11 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$L(A_2) = 9A_2 + 2A_2^T = 9A_2 + 2A_3$$

$$= \underbrace{0 \cdot A_1 + 9A_2 + 2A_3 + 0A_4}_{} \rightsquigarrow [L(A_2)]_E = \begin{pmatrix} 0 \\ 9 \\ 2 \\ 0 \end{pmatrix}$$

$$L(A_3) = 9A_3 + 2A_3^T = 9A_3 + 2A_2 \rightsquigarrow [L(A_3)]_E = \begin{pmatrix} 0 \\ 2 \\ 9 \\ 0 \end{pmatrix}$$

$$L(A_4) = 9A_4 + 2A_4^T = 9A_4 + 2A_4 = 11A_4$$

$$\rightsquigarrow [L(A_4)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 11 \end{pmatrix}.$$

$$[L]_E = \begin{pmatrix} 11 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 2 & 9 & 0 \\ 0 & 0 & 0 & 11 \end{pmatrix}.$$

9 $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. Find eigenvalues and bases of eigenspaces.

char poly $p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ -1 & 1 & -\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 + \lambda =$

$= \lambda(\lambda-1)^2$

$\rightsquigarrow$ Eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$

- Eigenspace for $\lambda_1 = 0$ is $N(A - \lambda_1 I) = N(A) \overset{\text{bring } A \text{ to RREF}}{=} N\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

  $\rightsquigarrow$ leading vars x_1, x_2, free var $x_3 = \alpha$

  $\Rightarrow$ $\begin{aligned} x_1 &= -\alpha \\ x_2 &= -\alpha \\ x_3 &= \alpha \end{aligned}$ $\rightsquigarrow N(A - \lambda_1 I) = \operatorname{Span}\left(\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right)$

- Eigenspace for $\lambda_2 = 1$ is $N(A - \lambda_2 I) = N(A - I)$

  $= N\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = N\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

  $\rightsquigarrow$ leading var: x_1, free vars $x_2 = \alpha$, $x_3 = \beta$

$\begin{aligned} x_1 &= \alpha - \beta \\ x_2 &= \alpha \\ x_3 &= \beta \end{aligned}$ $\quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\rightsquigarrow N(A - \lambda_2 I) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right)$.

(11) $A = \begin{pmatrix} 2 & -4 & 12 \\ 3 & -5 & 9 \\ 0 & 0 & -2 \end{pmatrix}$. Diagonalize!

First, find all eigenvalues and bases for eigenspaces.

Char poly $p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 & 12 \\ 3 & -5-\lambda & 9 \\ 0 & 0 & -2-\lambda \end{vmatrix} =$

$= \lambda^3 + 5\lambda + 8\lambda + 4 = (\lambda+1)(\lambda+2)^2$

$\rightsquigarrow$ Eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$.

- Eigenspace for $\lambda_1 = -1$ is $N(A - \lambda_1 I) = N(A + I)$

$= N\begin{pmatrix} 3 & -4 & 12 \\ 3 & -4 & 9 \\ 0 & 0 & -1 \end{pmatrix} = N\begin{pmatrix} 1 & -4/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}\right)$

- Eigenspace for $\lambda_2 = -2$ is $N(A - \lambda_2 I) = N(A + 2I)$

$= N\begin{pmatrix} 4 & -4 & 12 \\ 3 & -3 & 9 \\ 0 & 0 & 0 \end{pmatrix} = N\begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right)$

Hence $\xRightarrow{\text{Diag. Thm}}$ $P = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

How to make sure that you have 100% correct answer?

Calculate $P^{-1}AP$ and compare to D.

(12) $H = \text{Span}\left(\begin{pmatrix} 3 \\ 4 \\ -6 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -6 \\ 4 \\ -42 \end{pmatrix} \right)$ $\quad \underline{v} = \begin{pmatrix} -5 \\ 8 \\ -5 \\ -7 \end{pmatrix}$

$\underline{u}_1$ " $\quad$ $\underline{u}_2$ "

By the Best Approximation Thm., the answer is

$$\text{proj}_H \underline{v}.$$

Note:

$\underline{u}_1 \cdot \underline{u}_2 = 0$ so $(\underline{u}_1, \underline{u}_2)$ is an orthogonal basis for H.

if it weren't, we would have to find an orthogonal basis for H using Gram-Schmidt

Know: $\text{proj}_H \underline{v} = \left(\frac{v \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \left(\frac{v \cdot u_2}{u_2 \cdot u_2} \right) u_2 =$

$$= \frac{54}{62} u_1 + \frac{216}{1820} u_2 = \begin{pmatrix} 2.8503 \\ 2.7718 \\ -4.7511 \\ -5.8556 \end{pmatrix}.$$