Main Examination period 2024 – January – Semester A

MTH5112: Linear Algebra I

Examiners: R. Russo, I. Tomašić

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You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Part I: Multiple-choice questions

Question 1 [10 marks].
Determine whether the given set S is a subspace of the vector space V, and select those that are subspaces. [**10**]

(a) $V = \mathbb{R}^n$, and S is the set of solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ where A is a fixed $m \times n$ matrix;

(b) $V = C^2(\mathbb{R})$ (the space of twice continuously differentiable functions), and S is the subset of V consisting of those functions satisfying the differential equation $y'' - 4y' + 3y = 0$;

(c) $V = \mathbb{R}^2$, and S consists of all vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfying $x_1^2 - x_2^2 = 0$;

(d) V is the vector space of all real-valued functions defined on $\mathbb{R}$ and S is the subset of V consisting of those functions satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$;

(e) $V = P_n$ (the space of polynomials of degree up to n), and S is the subset of P_n consisting of those polynomials satisfying $p(t+1) = p(t) + 1$.

Question 2 [10 marks]. Select the true statements below. [**10**]

(a) There exists a proper subspace S of $\mathbb{R}^3$ such that $\text{Span}\,(S) = \mathbb{R}^3$.

(b) The space P_n of polynomials of degree up to n has a basis consisting of polynomials that all have degree n.

(c) There exist vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}$ span $\mathbb{R}^3$.

(d) A subset of a spanning set can sometimes form a linearly independent set.

(e) For all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in a vector space, $\mathbf{a} \in \text{Span}\,(\mathbf{b}, \mathbf{c})$ implies that $\mathbf{c} \in \text{Span}\,(\mathbf{a}, \mathbf{b})$.

Question 3 [10 marks]. Let $\mathbf{x}$, $\mathbf{y}$ be arbitrary vectors in a vector space, and suppose that $\mathbf{z} = 4\mathbf{x} + 3\mathbf{y}$ and $\mathbf{w} = -8\mathbf{x} - 6\mathbf{y} + 3\mathbf{z}$. Select true statements below. [**10**]

(a) $\text{Span}\,(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Span}\,(\mathbf{w}, \mathbf{x}, \mathbf{y})$;

(b) $\text{Span}\,(\mathbf{w}, \mathbf{z}) = \text{Span}\,(\mathbf{w}, \mathbf{x}, \mathbf{z})$;

(c) $\text{Span}\,(\mathbf{x}, \mathbf{y}) = \text{Span}\,(\mathbf{y}, \mathbf{z})$;

(d) $\text{Span}\,(\mathbf{w}, \mathbf{x}) = \text{Span}\,(\mathbf{w}, \mathbf{y}, \mathbf{z})$;

(e) $\text{Span}\,(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Span}\,(\mathbf{w}, \mathbf{z})$.

Question 4 [10 marks]. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 7 \\ 2 & 0 & x & -5 \\ -1 & 3 & -4 & 16 \end{pmatrix}.$$

Select the true statements below. [**10**]

(a) The rank of A is 3 for any value of x.

(b) The column space of A is 3-dimensional for any value of x.

(c) The nullity of A is 4 minus its rank.

(d) The rank of A^T is equal to the rank of A.

(e) The nullity of A^T is equal to the nullity of A.

Question 5 [10 marks]. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & 0 \\ 0 & -1 & 3 \end{pmatrix}.$$

Select the true statements below. [**10**]

(a) The vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of A.

(b) A has eigenvalue 0.

(c) The sum of all eigenvalues is 6.

(d) A is an orthogonal matrix.

(e) The determinant of A is 6.

Question 6 [10 marks]. Let P_3 be the vector space of all real polynomials in variable t of degree up to 2. Consider the linear transformation $D : P_3 \to P_3$ given by

$$D(p)(t) = (at + 1)\,\frac{dp(t)}{dt} + p(t)\ ,$$

where $p \in P_3$, $a \in \mathbb{R}$, and let A be the matrix associated to D with respect to the basis $(t^2, t, 1)$. Select the true statements below. [**10**]

(a) The determinant of A vanishes for $a = 0$.

(b) The determinant of A vanishes for $a = -\frac{1}{2}$.

(c) A is diagonalisable for all values of a.

(d) $A + A^T$ is diagonalisable.

(e) The rank of A is 2 when $a = -1$.

Question 7 [10 marks]. Suppose that

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Which of the following vectors is a least-squares solution to $A\mathbf{x} = \mathbf{b}$? [**10**]

(a) $\mathbf{x} = \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 2 \end{pmatrix}$;

(b) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ \frac{1}{3} \end{pmatrix}$;

(c) $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$;

(d) $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$;

(e) $\mathbf{x} = \begin{pmatrix} \frac{2}{3} \\ 0 \\ 3 \end{pmatrix}$.

Part II: Open-ended questions

Question 8 [7 marks]. Let V be a 4-dimensional vector space and let $L : V \to V$ be a linear transformation such that

$$L^4 = \mathbf{0} \text{ and } L^3 \neq \mathbf{0},$$

where $L^n = \underbrace{L \circ \cdots \circ L}_{n \text{ times}}$ denotes the n-fold composite of L with itself. Prove that there exists a basis B for V such that the matrix of L with respect to the basis B is

$$[L]_B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Justify all your claims and state precisely any theorems you use. [**7**]

Hint. Since $L^3 \neq \mathbf{0}$, there exists a vector $\mathbf{v} \in V$ such that $L^3(\mathbf{v}) \neq \mathbf{0}$. Consider the set

$$\{\mathbf{v}, L(\mathbf{v}), L^2(\mathbf{v}), L^3(\mathbf{v})\}.$$

Question 9 [8 marks]. Suppose that vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent in a vector space V, and let $\mathbf{w} \in V$ be another vector such that

$$\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_n + \mathbf{w}$$

are linearly dependent. Prove that

$$\mathbf{w} \in \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

Justify all your claims and state precisely any theorems you use. [**8**]

Question 10 [15 marks]. Consider the vector space $\mathbb{R}^{2\times 2}$ of real 2×2 matrices.

(a) Check that the ordered set $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is a basis for $\mathbb{R}^{2\times 2}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[**3**]

(b) Consider the basis $C = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4)$, with

$$\mathbf{w}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Write down the transition matrix $P_{B,C} = [\mathrm{id}]^B_C$ from the basis B to the basis C. [**4**]

(c) Check that the columns of $P_{B,C}$ form an orthogonal set of vectors in $\mathbb{R}^4$. Rescale those vectors to obtain an orthonormal set. [**4**]

(d) Consider a vector $\mathbf{w} \in \mathbb{R}^{2\times 2}$ and let $\mathbf{a} = [\mathbf{w}]_C \in \mathbb{R}^4$. Calculate the norm of $\mathbf{a}$ according to the standard scalar product in $\mathbb{R}^4$.

Show that $\mathbf{w}$ has the same norm with respect to the scalar product on $\mathbb{R}^{2\times 2}$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathrm{Tr}(\mathbf{v}^T \mathbf{w}),$$

for $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{2\times 2}$, where Tr is the trace (the sum of the diagonal elements of a matrix). [**4**]

End of Paper.