MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 10

Exercise (*) 1. Solve WeBWork **Set 10** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023/`.

Log in with your 'ah***' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. The Orthogonal Decomposition Theorem (7.26 from lectures) says the following: if H is a subspace of $\mathbb{R}^n$ then every vector $\mathbf{v} \in \mathbb{R}^n$ can be written *uniquely* in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}, \quad \text{where} \quad \hat{\mathbf{y}} \in H \text{ and } \mathbf{z} \in H^\perp,$$

and, furthermore, if $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is an orthogonal basis for H then

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{v}_r}{\mathbf{v}_r \cdot \mathbf{v}_r}\right)\mathbf{v}_r. \tag{1}$$

Proceed as follows to prove the Orthogonal Decomposition Theorem:

(a) Explain why we can *define* the vector $\hat{\mathbf{y}}$ using equation (1). In other words, why do we know that H actually *has* an orthogonal basis?
(b) Now *define* the vector $\mathbf{z}$ by setting $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Prove that $\mathbf{z}$ is an element of $H^\perp$.
(c) It now remains to show that the decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ of $\mathbf{y}$ into a vector in H plus a vector in $H^\perp$ is *unique*. To prove this, assume that we can also write

$$\mathbf{y} = \hat{\mathbf{y}}_1 + \mathbf{z}_1 \quad \text{for some} \quad \hat{\mathbf{y}}_1 \in H \text{ and } \mathbf{z}_1 \in H^\perp,$$

and show that this implies that $\hat{\mathbf{y}}_1 = \hat{\mathbf{y}}$ and $\mathbf{z}_1 = \mathbf{z}$. (Hint: can you prove that the vector $\hat{\mathbf{y}} - \hat{\mathbf{y}}_1$ must be an element of both H and $H^\perp$?)

Exercise 3. Prove Corollary 7.28 (of the Orthogonal Decomposition Theorem) from lectures: if H is a subspace of $\mathbf{R}^n$, then

$$(H^\perp)^\perp = H.$$

Exercise 4. Consider the vectors $\mathbf{y} = (-1, 7)^T$ and $\mathbf{u} = (1, 3)^T$ in $\mathbb{R}^2$.

(a) Compute the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto the subspace $H = \text{span}(\mathbf{u})$.
(b) Write down the vector $\mathbf{y} - \hat{\mathbf{y}}$ and verify that it really is orthogonal to $\mathbf{u}$.
(c) Give a geometric interpretation of the quantity $||\mathbf{y} - \hat{\mathbf{y}}||$.

Exercise 5. Consider the vectors $\mathbf{y} = (6, -1, 8)^T$, $\mathbf{u}_1 = (1, 2, 1)^T$ and $\mathbf{u}_2 = (-3, 1, 1)^T$ in $\mathbf{R}^3$, and let $H = \text{span}(\mathbf{u}_1, \mathbf{u}_2)$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set (and hence a basis for H), and find the orthogonal decomposition of $\mathbf{y}$ with respect to the subspace H, i.e. write $\mathbf{y}$ as the sum of a vector in H and a vector in $H^\perp$.

Exercise 6. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{x}_1 = (1, 0, 1, 0)^T, \quad \mathbf{x}_2 = (3, 0, 1, 1)^T, \quad \mathbf{x}_3 = (-2, 1, 4, -3)^T.$$

(a) Show that the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent but *not* orthogonal.

(b) Use the Gram–Schmidt process to construct an orthogonal basis for the subspace $H = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of $\mathbb{R}^4$.

(c) Find the orthogonal decomposition of the vector $\mathbf{y} = (2, 1, 4, -4)^T$ with the respect to the subspace H (i.e. write $\mathbf{y}$ as the sum of a vector in H and a vector in $H^\perp$), and thereby determine the best approximation to $\mathbf{y}$ by a vector in H.

Exercise 7. Solve the least squares problem for the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}.$$

That is, write down the corresponding normal equations and determine the set of least squares solutions.