Week 12

Applications of Linear Algebra

Spectral Theorem

Recall : • $A \in \mathbb{R}^{n \times n}$ is symmetric if $A^T = A$

• $Q \in \mathbb{R}^{n \times n}$ is orthogonal if $Q^T Q = I$

Th (Spectral Theorem for symmetric matrices)

If $A \in \mathbb{R}^{n \times n}$ is symmetric, then :

(i) All eigenvalues of A are real

(ii) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

(iii) A can be diagonalized by an orthogonal matrix, i.e., there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$, s.t.

$$Q^T A Q = D,$$

where D is diagonal.

Remark To diagonalize a symmetric A by an orthogonal matrix Q :

- find a basis for each eigenspace and use Gram-Schmidt to make it orthonormal
- by Spectral Th, orthonormal bases of all eigenspaces jointly form an orthonormal basis for $\mathbb{R}^n$
- form the matrix Q using the orthonormal basis vectors as columns.

Applications of Linear Algebra

① Solving linear recursions / difference equations

<u>Example</u>: Fibonacci sequence

$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_n + F_{n+1}$

$\rightsquigarrow 0, 1, 1, 2, 3, 5, 8, \dots \qquad F_n = ?$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \dots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma \underline{x} = A \underline{x}, \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix};$$

SHIFT

Golden ratio

$$p_A(\lambda) = \lambda^2 - \lambda - 1, \qquad \text{eigenvalues } \lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\text{eigenvectors } \underline{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$$

Diagonalize A:

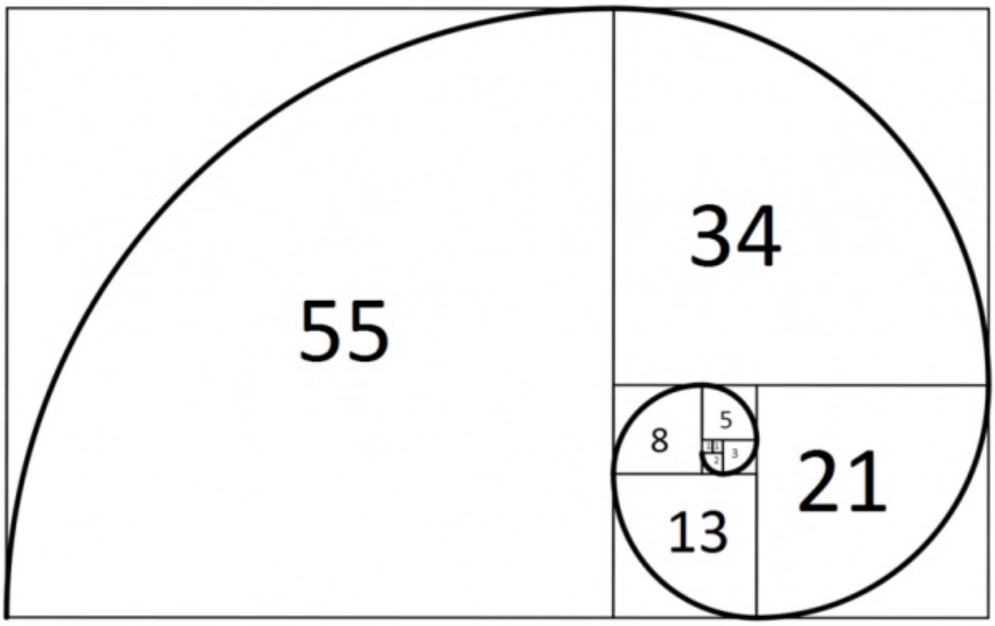
$$P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

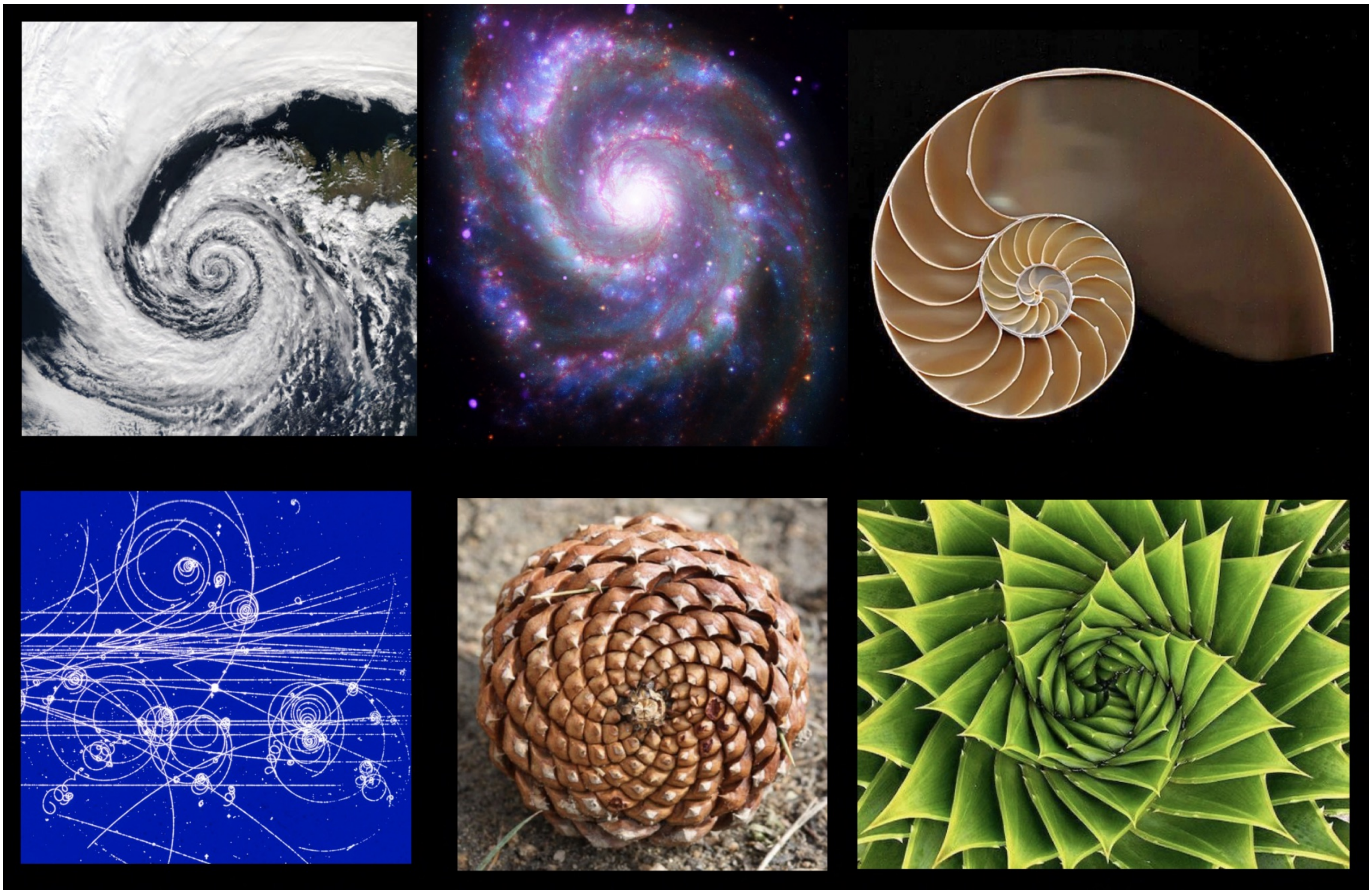
$$\Rightarrow A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}, \qquad P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

$$\Rightarrow A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}$$

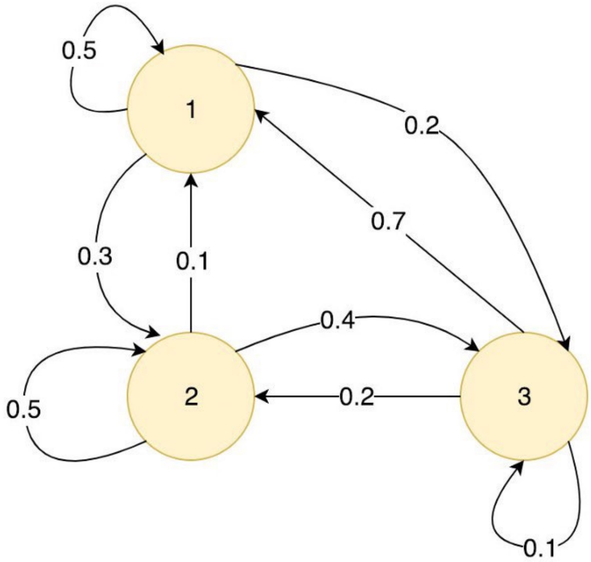
and
$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \boxed{F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}}$$

Golden ratio in nature :





(2) Markov Chains $\rightsquigarrow$ finance



Markov process

$$\begin{array}{cc} & \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} 0.5 & 0.1 & 0.7 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.1 \end{bmatrix} \end{array}$$

Transition matrix

③ Linear differential equations

$$\underline{y}' = A\underline{y}, \text{ where } \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ \underline{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$$

The solution set consists of

$$y \in \left(C^1(\mathbb{R})\right)^n \text{ that satisfy the equation,}$$

and it has a structure of a vector space:

if $\quad \underline{u}' = A\underline{u}$ and $\underline{v}' = A\underline{v}$, then

$$(\underline{u}+\underline{v})' = \underline{u}' + \underline{v}' = A\underline{u} + A\underline{v} = A(\underline{u}+\underline{v})$$

and $\quad (\lambda \underline{u})' = \lambda \underline{u}' = \lambda A \underline{u} = A(\lambda \underline{u})$.

Ex
$$\begin{aligned} y_1' &= 2y_1 - 3y_2 \\ y_2' &= y_1 - 2y_2 \end{aligned} \rightsquigarrow \underline{y}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \underline{y}$$

$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$ is diagonalizable : $A = PDP^{-1}$, $P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Change of variables : $\underline{y} = P\underline{w}$ (i.e. $\underline{w} = P^{-1}\underline{y}$)

$$y' = PDP^{-1}\underline{y} \Rightarrow P^{-1}y' = DP^{-1}\underline{y} \Rightarrow (P^{-1}\underline{y})' = D(P^{-1}\underline{y})$$

$$\Rightarrow \quad \underline{w}' = D\underline{w}, \text{ i.e. } \quad \begin{aligned} w_1' &= w_1 \\ w_2' &= -w_2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} w_1(t) &= C_1 e^{t} \\ w_2(t) &= C_2 e^{-t} \end{aligned}$$

$$\Rightarrow \underline{y} = P\underline{w} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{t} \\ C_2 e^{-t} \end{pmatrix}$$

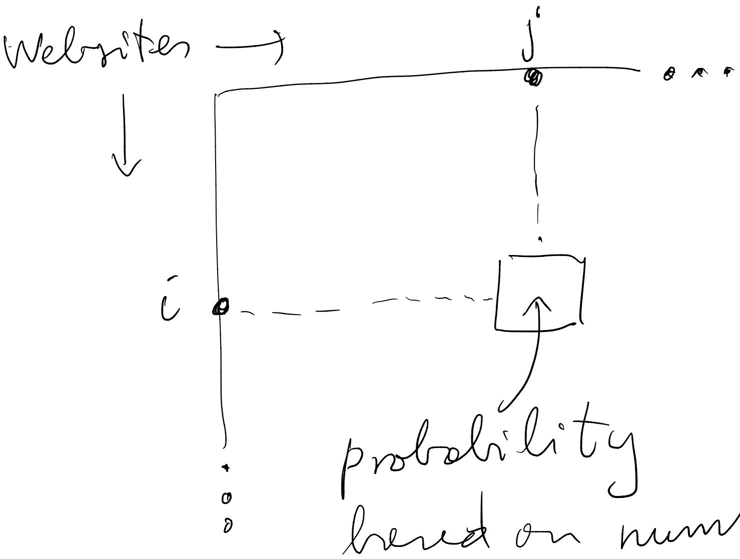
$$\Rightarrow \begin{aligned} y_1 &= 3C_1 e^{t} + C_2 e^{-t} \\ y_2 &= C_1 e^{t} + C_2 e^{-t}. \end{aligned}$$

"general" solution

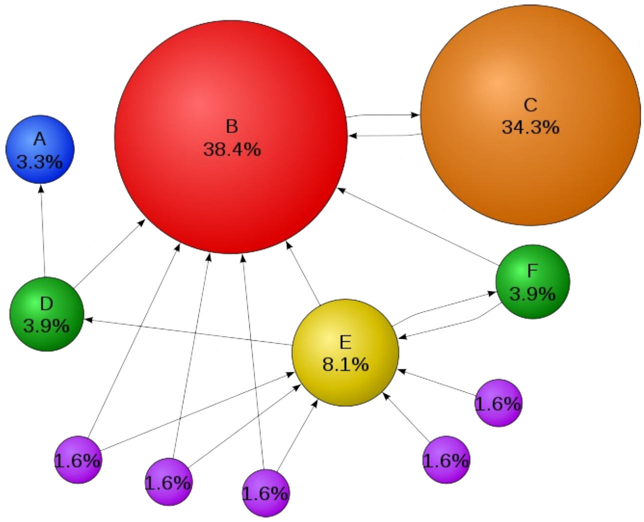
Solution set has dimension 2.

④ Google PageRank algorithm

Incidence matrix of the internet



probability based on number of links from i to j.



Calculate approximate eigenvectors.

⑤ Networks: - electrical
- social
- transport
- disease transmission

⑥ Image/signal processing

(7) 3D Computer Graphics: - CGI effects in films
- computer games
- design

Work in $\mathbb{R}^3$.

Geometric transformations: - rotations
- reflections
- scaling, etc

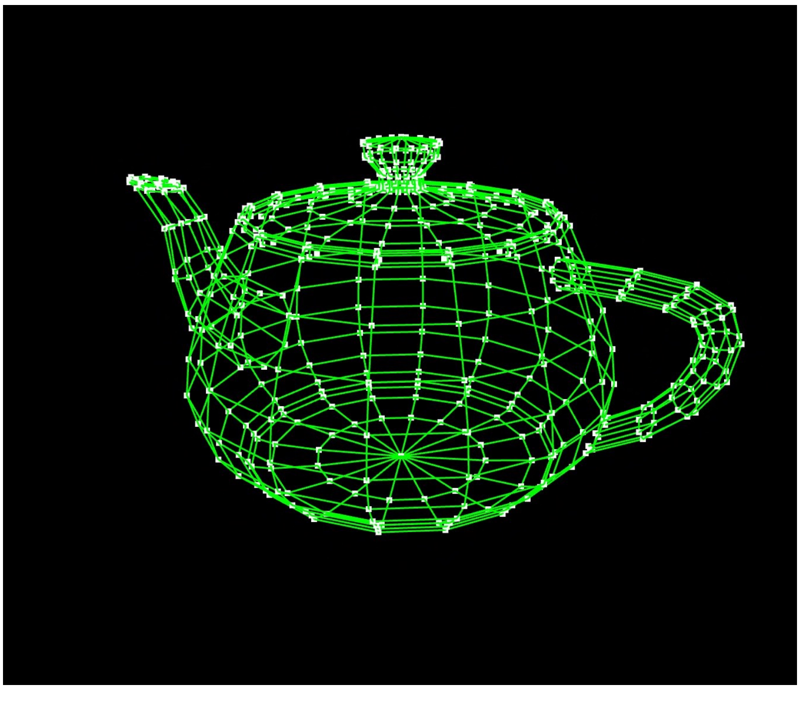
are in fact linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$

$$\underline{x} \longmapsto A\underline{x}$$

for a suitable A:

- rotation if A is orthogonal with $\det(A) = 1$
- reflection — — — — $\det(A) = -1$

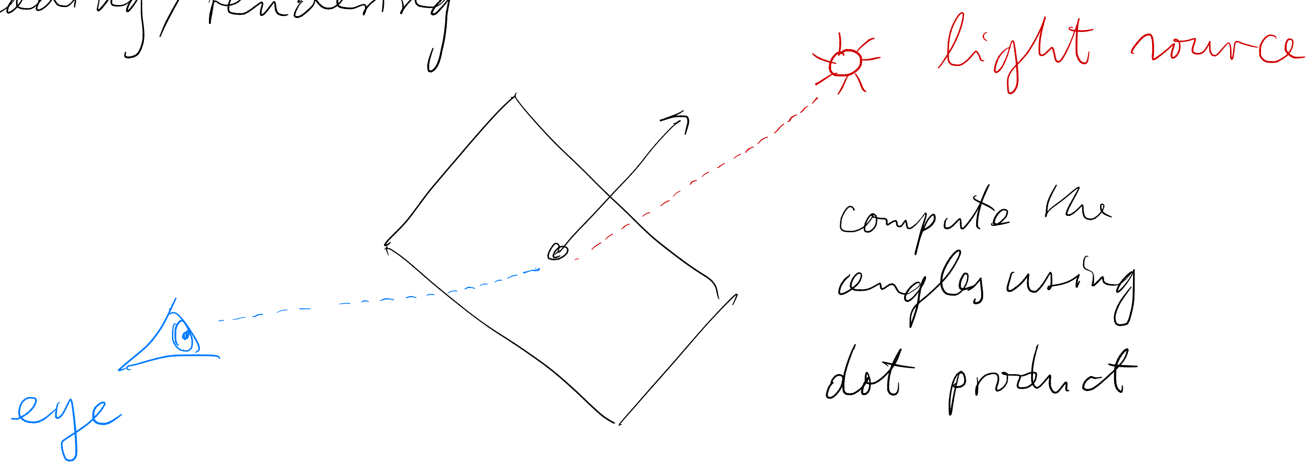
Representing shapes:



Points are represented by $\underline{x}_1, \underline{x}_2 \dots \underline{x}_n \in \mathbb{R}^3$

$\rightsquigarrow$ matrix of coordinates $X \in \mathbb{R}^{3\times n}$

Shading / rendering



compute the angles using dot product

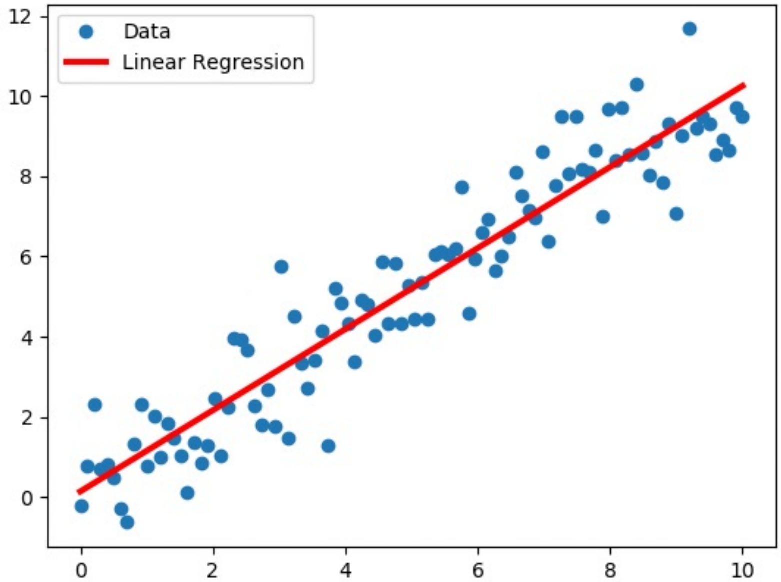
Animate: Perform a transformation

$$X \longmapsto AX$$

and draw / render the new shape incrementally (A's vary in time).

(8) Machine learning - neural networks

(9) Data science / analysis



linear regression ⟷ least squares method

(10) Coding Theory (QMUL module)

- linear codes are vector spaces over finite fields $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

(11) Cryptography (QMUL module)