MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 8 — SOLUTIONS

Exercise ($*$) 1. The solutions will appear on WeBWork after CW8 due date.

Exercise 2. (a) 0 is an eigenvalue of A if and only if $\det(A) = 0$, which, by the Invertible Matrix Theorem, is the case if and only if A is not invertible.

(b) Suppose that A is invertible. If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Moreover, by part (a), $\lambda \neq 0$. To show that λ^{-1} is an eigenvalue of A^{-1}, we must show that $A^{-1}\mathbf{w} = \lambda^{-1}\mathbf{w}$ for some nonzero vector $\mathbf{w}$. In fact, we will show that we can take $\mathbf{w} = \mathbf{v}$. Multiplying both sides of the equation $A\mathbf{v} = \lambda\mathbf{v}$ on the left by $\lambda^{-1}A^{-1}$, we obtain

$$\lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}\lambda A^{-1}\mathbf{v},$$

so

$$\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v},$$

which is what we wanted. Conversely, suppose that λ^{-1} is an eigenvalue of A with eigenvector $\mathbf{v}$, that is,

$$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}\,.$$

Multiplying both sides of this equation on the left by λA, we obtain

$$\lambda A A^{-1}\mathbf{v} = \lambda\lambda^{-1}A\mathbf{v}\,,$$

so

$$\lambda\mathbf{v} = A\mathbf{v},$$

i.e. λ is an eigenvalue of A.

(c) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then, as usual,

$$A\mathbf{v} = \lambda\mathbf{v}.$$

To show that λ^n is an eigenvalue of A^n for every positive integer n, we must show that $A^n\mathbf{w} = \lambda^n\mathbf{w}$ for some nonzero vector $\mathbf{w}$. In fact, we will show that we can take $\mathbf{w} = \mathbf{v}$. That is, we will show that

$$A^n\mathbf{v} = \lambda^n\mathbf{v}$$

for every positive integer n. This is true for $n = 2$ because

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}.$$

Now let's finish the proof using induction. Suppose that we have proved the result for an integer $n-1$, i.e. we know that $A^{n-1}\mathbf{v} = \lambda^{n-1}\mathbf{v}$. We must now prove it for the integer n, which we can do as follows:

$$A^n\mathbf{v} = A(A^{n-1}\mathbf{v}) = A(\lambda^{n-1}\mathbf{v}) = \lambda^{n-1}(A\mathbf{v}) = \lambda^{n-1}(\lambda\mathbf{v}) = \lambda^n\mathbf{v}.$$

Exercise 3. (a) Since

$$\begin{pmatrix} 4 & 3 & 3 \\ -4 & -3 & -4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix},$$

the vector $(3, -4, 2)^T$ is indeed an eigenvector of A, corresponding to the eigenvalue 2.

(b) A straightforward but somewhat tedious calculation shows that the characteristic polynomial of A, call if p, is

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2.$$

The eigenvalues of A are the roots of p, i.e. the solutions of $p(\lambda) = 0$. One way to find the roots is to observe from (a) that $\lambda = 2$ is a root of p, so that we can take out a factor $\lambda - 2$ and then use polynomial long division. Regardless of how you choose to do it, you should find that

$$p(\lambda) = -(\lambda - 2)(\lambda^2 - 2\lambda + 1) = -(\lambda - 2)(\lambda - 1)^2.$$

Hence, the eigenvalues of A are 1 and 2. To find the eigenspaces, we now just solve (in the usual way) the linear system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

for each of the (two) eigenvalues λ that we found above. You should be very good at solving linear systems by now, so I won't write down the calculations, but you should find that the eigenspace corresponding to $\lambda = 1$ is

$$N(A - I) = \mathrm{Span}\,(\mathbf{v}_1, \mathbf{v}_2), \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and that the eigenspace corresponding to $\lambda = 2$ is

$$N(A - 2I) = \mathrm{Span}\,(\mathbf{v}_3), \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}.$$

(c) From (b) we see that A has three linearly independent eigenvectors (if this is not clear then observe that the two eigenvectors corresponding to $\lambda = 1$ are clearly linearly independent, and recall Theorem 5.7 in the lecture notes which guarantees that eigenvectors corresponding to different eigenvalues are always linearly independent). Hence, the Diagonalisation Theorem (5.8 in the lecture notes) tells us that A is diagonalisable. Moreover, the theorem actually gives us a matrix P that diagonalises A: it's just the matrix whose columns are the eigenvectors of A. That is, if we now define

$$P = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix},$$

then P is invertible and

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We can check whether we have made any mistakes by calculating

$$AP = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -3 & -4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix}$$

and

$$PD = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix},$$

which show that $AP = PD$ as required.

(d) Since $P^{-1}AP = D$ it follows that

$$A = PDP^{-1},$$

and hence (as explained in more generality in lectures) that

$$A^5 = PD^5P^{-1}.$$

Therefore,

$$A^5 = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 2^5 \end{pmatrix} \begin{pmatrix} 4 & 5 & 4 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 94 & 93 & 93 \\ -124 & -123 & -124 \\ 62 & 62 & 63 \end{pmatrix}.$$

Exercise 4. As in the previous exercise, I'm going omit some of the details of various routine calculations (of determinants, row operations, back substitution) in this exercise. You should be very good by now at computing determinants and solving systems of linear equations via Gaussian elimination/back substitution, so you shouldn't need me to write down all the details.

(a) Let

$$A = \begin{pmatrix} 6 & 4 & 2 \\ -7 & -6 & -5 \\ 4 & 4 & 4 \end{pmatrix}.$$

Then

$$\det(A - \lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(\lambda^2 - 4\lambda + 4) = -\lambda(\lambda - 2)^2,$$

so the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$. The corresponding eigenspaces, i.e. the nullspaces $N(A - \lambda I)$ of the matrices $A - \lambda I$ (i.e. the sets of solutions of the linear systems $(A - \lambda I)\mathbf{x} = \mathbf{0}$), are obtained as follows. For the eigenvalue $\lambda_1 = 0$ we have

$$A - \lambda_1 I = A \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$N(A) = \mathrm{Span}\,(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = 2$ we have

$$A - \lambda_2 I = A - 2I = \begin{pmatrix} 4 & 4 & 2 \\ -7 & -8 & -5 \\ 4 & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$N(A - 2I) = \mathrm{Span}\,(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Since the 3×3 matrix A has only 2 linearly independent eigenvectors, we conclude that A is not diagonalisable.

(b) Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since A is (upper) triangular, we can immediately see that its eigenvalues are 2 and 3. The corresponding eigenspaces are

$$N(A-2I) = N\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) = \mathrm{Span}\,(\mathbf{v}_1, \mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$N(A-3I) = N\left(\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \mathrm{Span}\,(\mathbf{v}_3, \mathbf{v}_4) \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the 4×4 matrix A has 4 linearly independent eigenvectors, A is diagonalisable and is diagonalised by the matrix

$$P = (\mathbf{v}_1 \; \mathbf{v}_2 \; \mathbf{v}_3 \; \mathbf{v}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e.

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(c) This is very similar to (b). Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since A is (upper) triangular, we see immediately that its eigenvalues are 2 and 3. The corresponding eigenspaces are

$$N(A-2I) = \mathrm{Span}\,(\mathbf{v}_1, \mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and

$$N(A-3I) = \mathrm{Span}\,(\mathbf{v}_3) \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since the 4×4 matrix A has only 3 linearly independent eigenvectors, it is not diagonalisable.