MTH 5112/5212

COURSEWORK 7 SOLUTIONS

(2) To show that $\mathrm{im}(L) = \mathrm{Span}(L(\underline{v}_1) \dots L(\underline{v}_n))$, it is enough to verify that $\mathrm{im}(L) \subseteq \mathrm{Span}(L(\underline{v}_1), \dots, L(\underline{v}_n))$, because the reverse inclusion is clear.

Hence, let $\underline{w} \in \mathrm{im}(L)$ be arbitrary. By defn. of $\mathrm{im}(L)$, there is a $\underline{v} \in V$ s.t. $\underline{w} = L(\underline{v})$. Since $V = \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n)$, we can write $\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

But then
$$\underline{w} = L(\underline{v}) = L(\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n) = \alpha_1 L(\underline{v}_1) + \dots + \alpha_n L(\underline{v}_n) \in \mathrm{Span}(L(\underline{v}_1), \dots, L(\underline{v}_n)).$$
$\square$

(3) (1-2) $\ker(L) = \{\underline{x} \in \mathbb{R}^3 : L(\underline{x}) = \underline{0}\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} 16x_1 - 20x_2 - 16x_3 \\ -10x_2 \\ 48x_1 \quad -48x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$, i.e.

$\ker(L)$ is the set of solutions of the homogeneous system of lin. equations

$$\begin{aligned} 16x_1 - 20x_2 - 16x_3 &= 0 \\ -10x_2 &= 0 \\ 48x_1 \qquad - 48x_3 &= 0 \end{aligned}$$

Solve it using G-J elimination:
$$\left(\begin{array}{ccc|c} 16 & -20 & -16 & 0 \\ 0 & -10 & 0 & 0 \\ 48 & 0 & -48 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 16 & 0 & -16 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Free variable: $x_3 = \alpha$, $x_2 = 0$, $x_1 = x_3 = \alpha$
$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \ker(L) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right),$$

so basis for $\ker(L)$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$, $\dim(\ker(L)) = \mathrm{nul}(L) = 1$.

By Exercise 1, $\mathrm{im}(L) = \mathrm{Span}(L(\underline{e}_1), L(\underline{e}_2), L(\underline{e}_3))$, where $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is the standard basis for $\mathbb{R}^3$.

Thus, $\mathrm{im}(L) = L(\mathbb{R}^3) = \mathrm{Span}\left(\begin{pmatrix} 16 \\ 0 \\ 48 \end{pmatrix}, \begin{pmatrix} -20 \\ -10 \\ 0 \end{pmatrix}, \begin{pmatrix} -16 \\ 0 \\ -48 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right)$

lin. indep.

So basis for $\mathrm{im}(L)$ is $\left\{\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right\}$, $\dim(\mathrm{im}(L)) = \mathrm{rank}(L) = 2$.

Finally, $\mathrm{rank}(L) + \mathrm{im}(L) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$, in accordance with the Rank-Nullity theorem for linear maps.

(3) The matrix A associated with L has columns $L(\underline{e}_1), L(\underline{e}_2), L(\underline{e}_3)$, where $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is the standard basis for $\mathbb{R}^3$. Hence,
$$A = \begin{pmatrix} 16 & -20 & -16 \\ 0 & -10 & 0 \\ 48 & 0 & -48 \end{pmatrix}.$$