MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 6 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW6 due date.

Exercise 2. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis for $L \cap M$. In particular, this set is linearly independent in both L and M, so, by a result from lectures, it can be completed to bases of L and M, i.e., we can find $\mathbf{u}_1, \ldots, \mathbf{u}_s$ in L and $\mathbf{w}_1, \ldots, \mathbf{w}_t$ in M so that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\}$ is a basis for L, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{w}_1, \ldots, \mathbf{w}_t\}$ is a basis for M.

It suffices to show that

$$B = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{w}_1, \ldots, \mathbf{w}_t\}$$

is a basis for $L + M$, since it would then follow that

$$\dim(L + M) + \dim(L \cap M) = (r + s + t) + r = (r + s) + (r + t) = \dim(L) + \dim(M).$$

To see that B is linearly independent, assume that

$$\sum_{i=1}^{r} \alpha_i \mathbf{v}_i + \sum_{j=1}^{s} \beta_j \mathbf{u}_j + \sum_{k=1}^{t} \gamma_k \mathbf{w}_k = \mathbf{0}.$$

Then we have

$$\sum_{i=1}^{r} \alpha_i \mathbf{v}_i + \sum_{j=1}^{s} \beta_j \mathbf{u}_j = -\sum_{k=1}^{t} \gamma_k \mathbf{w}_k,$$

where the left hand side is a vector in L, and the right hand side is a vector in M, hence both are actually in $L \cap M$. Thus, there exist some scalars δ_i so that

$$-\sum_{k=1}^{t} \gamma_k \mathbf{w}_k = \sum_{i=1}^{r} \delta_i \mathbf{v}_i,$$

i.e.,

$$\sum_{i=1}^{r} \delta_i \mathbf{v}_i + \sum_{k=1}^{t} \gamma_k \mathbf{w}_k = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{w}_1, \ldots, \mathbf{w}_t\}$ is a basis for M, it is linearly independent and it follows that $\delta_i = 0$ for $i = 1, \ldots, r$ and $\gamma_k = 0$ for $k = 1, \ldots, t$. By substituting $\gamma_k = 0$ in the above, we see that

$$\sum_{i=1}^{r} \alpha_i \mathbf{v}_i + \sum_{j=1}^{s} \beta_j \mathbf{u}_j = \mathbf{0},$$

so using the fact that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{u}_1, \ldots, \mathbf{u}_s\}$ is a basis for L, we deduce that $\alpha_i = 0$ for $i = 1, \ldots, r$ and $\beta_j = 0$ for $j = 1, \ldots, s$.

To summarise, all α_i, β_j and γ_k must be 0, and we have proved linear independence of B.

To see that B is spanning set for $L + M$, note that an arbitrary vector $\mathbf{x} \in L + M$ can be written as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in L$, and $\mathbf{z} \in M$. These vectors can be expressed in terms of bases as

$$\mathbf{y} = \sum_{i=1}^{r} \alpha_i \mathbf{v}_i + \sum_{j=1}^{s} \beta_j \mathbf{u}_j, \text{ and } \mathbf{z} = \sum_{i=1}^{r} \alpha'_i \mathbf{v}_i + \sum_{k=1}^{t} \gamma_k \mathbf{w}_k$$

for some $\alpha_i, \alpha'_i, \beta_j, \gamma_k$. Thus,

$$\mathbf{x} = \mathbf{y} + \mathbf{z} = \sum_{i=1}^{r} (\alpha_i + \alpha'_i)\mathbf{v}_i + \sum_{j=1}^{s} \beta_j \mathbf{u}_j + \sum_{k=1}^{t} \gamma_k \mathbf{w}_k,$$

showing that

$$\mathbf{x} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{w}_1, \ldots, \mathbf{w}_r) = \mathrm{Span}(B),$$

and we are done.

Exercise 3. First put A to row echelon form:

$$\begin{aligned} A &= \begin{pmatrix} \mathbf{1} & -1 & 3 & \mathbf{1} & 2 \\ \mathbf{2} & -2 & 6 & \mathbf{3} & 0 \\ \mathbf{3} & -3 & 9 & \mathbf{4} & 2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 & -4 \end{pmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \\ &\sim \begin{pmatrix} \mathbf{1} & -1 & 3 & 1 & 2 \\ 0 & 0 & 0 & \mathbf{1} & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \end{matrix}. \end{aligned}$$

Call the final matrix U. When regarded as elements of $\mathbb{R}^5$ (i.e. as column vectors), the nonzero rows of U form a basis for $\mathrm{row}(A)$, so

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{pmatrix} \right\}$$

is a basis for $\mathrm{row}(A)$. Now look at the columns of U. The first and fourth columns contain the leading 1s, and a basis for $\mathrm{col}(A)$ is obtained by taking the corresponding columns *of the original matrix* A (*not* of the REF matrix U). Therefore,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$$

is a basis for $\mathrm{col}(A)$ (this is highlighted in red/boldface in the above calculation). In order to determine a basis for $N(A)$ we solve $U\mathbf{x} = \mathbf{0}$ in the usual way (using back substitution): setting $x_2 = \alpha$, $x_3 = \beta$, and $x_5 = \gamma$, we find $x_4 = 4x_5 = 4\gamma$, and $x_1 = x_2 - 3x_3 - x_4 - 2x_5 = \alpha - 3\beta - 6\gamma$. Thus, every solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\begin{pmatrix} \alpha - 3\beta - 6\gamma \\ \alpha \\ \beta \\ 4\gamma \\ \gamma \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{x}_1} + \beta \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{x}_2} + \gamma \underbrace{\begin{pmatrix} -6 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}}_{=\mathbf{x}_3},$$

for some α, β, γ, and so $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for $N(A)$.

Since the bases for $\text{row}(A)$ and $N(A)$ found above have two and three elements, respectively, we have

$$\text{rank}\,(A) = 2 \quad \text{and} \quad \text{null}\,(A) = 3.$$

Hence, $\text{rank}\,(A) + \text{null}\,(A) = 3 + 2 = 5$, so the Rank–Nullity Theorem holds for the matrix A.

Exercise 4. By the Invertible Matrix Theorem, A is invertible if and only if $N(A) = \{\mathbf{0}\}$, i.e. A is invertible if and only if $\text{null}\,(A) = 0$. By the Rank–Nullity Theorem, $\text{rank}\,(A) = n - \text{null}\,(A)$, so A is invertible if and only if $\text{rank}\,(A) = n$.

Exercise 5. (a) Yes, e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 1 and nullity 2. (If this is not clear then you should check it!)

(b) No, this is not possible. The Rank–Nullity Theorem says that for every 3×4 matrix A, we have $\text{rank}\,(A) + \text{null}\,(A) = 4$, so no 3×4 matrix can have rank 2 and nullity 1 because these numbers add up to $3 \neq 4$.