MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 5 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW5 due date.

Exercise 2. (a) Because (as you can check)

$$\begin{vmatrix} 1 & 0 & -3 \\ -2 & 1 & 6 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0 \quad \text{and} \quad \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{vmatrix} = -3 \neq 0,$$

Theorem 4.12 from lectures tells us that $\mathcal{B}_1$ and $\mathcal{B}_2$ are both linearly independent. Since both sets contain 3 vectors in the 3-dimensional vector space $\mathbb{R}^3$, both are therefore bases for $\mathbb{R}^3$.

(b) By definition, the transition matrix from $\mathcal{B}_2$ to the standard basis of $\mathbb{R}^3$ is the matrix whose columns are the vectors in $\mathcal{B}_2$, i.e.

$$P_{\mathcal{B}_2} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

(c) Notice that the question asks you to determine the transition matrix from the standard basis to $\mathcal{B}_1$, which is the *inverse* of the transition matrix $P_{\mathcal{B}_1}$ from $\mathcal{B}_1$ to the standard basis:

$$P_{\mathcal{B}_1}^{-1} = \begin{pmatrix} 1 & 0 & -3 \\ -2 & 1 & 6 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & -3 & 3 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix}.$$

(I'll omit the details of how to compute this inverse using the Gauss–Jordan algorithm, because you should know how to do that by now.) The transition matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$ is therefore

$$P_{\mathcal{B}_1}^{-1} P_{\mathcal{B}_2} = \begin{pmatrix} -17 & -7 & -2 \\ 5 & 4 & 2 \\ -6 & -3 & -1 \end{pmatrix}.$$

(d) We have

$$[\mathbf{x}]_{\mathcal{B}_1} = P_{\mathcal{B}_1}^{-1} P_{\mathcal{B}_2} [\mathbf{x}]_{\mathcal{B}_2} = \begin{pmatrix} -17 & -7 & -2 \\ 5 & 4 & 2 \\ -6 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Exercise 3. (a) We know that P_2 has dimension 3, so we just need to check that the three vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent. We must therefore assume that

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0},$$

and show that this assumption implies that $c_1 = c_2 = c_3 = 0$. For each $t \in \mathbb{R}$ we have

$$\begin{aligned} 0 &= c_1\mathbf{p}_1(t) + c_2\mathbf{p}_2(t) + c_3\mathbf{p}_3(t) \\ &= c_1(t^2 - 4t + 2) + c_2(t + 3) + c_3 \cdot 1 \\ &= c_1t^2 + (c_2 - 4c_1)t + (2c_1 + 3c_2 + c_3), \end{aligned}$$

and so we obtain the following linear system for the unknowns c_1, c_2, c_3:

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ -4c_1 + c_2 \qquad &= 0 \\ 2c_1 + 3c_2 + c_3 &= 0. \end{aligned}$$

Although we could now use Gaussian elimination, it is reasonably clear that this system has only the trivial solution: the first equation says that $c_1 = 0$, then the second equation gives $c_2 = -4c_1 = 0$, and similarly the third equation then gives $c_3 = 0$. Therefore, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent (and hence a basis for P_2, as explained above).

(b) By definition, $[\mathbf{p}]_{\mathcal{B}} = (-1, 3, 2)^T$ means that

$$\begin{aligned} \mathbf{p}(t) &= -\mathbf{p}_1(t) + 3\mathbf{p}_2(t) + 2\mathbf{p}_3(t) \\ &= -t^2 + 7t + 9. \end{aligned}$$

(c) We must find scalars c_1, c_2, c_3 such that $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{q}$. Using similar working to that in part (a), we see that we must therefore find c_1, c_2, c_3 such that

$$c_1t^2 + (c_2 - 4c_1)t + (2c_1 + 3c_2 + c_3) = -t^2 + 6$$

for all $t \in \mathbb{R}$. Comparing powers of t now gives us the following system of equations for c_1, c_2, c_3:

$$\begin{aligned} c_1 \qquad\qquad &= -1 \\ -4c_1 + c_2 \qquad &= 0 \\ 2c_1 + 3c_2 + c_3 &= 6. \end{aligned}$$

The (unique) solution is $c_1 = -1$, $c_2 = -4$ and $c_3 = 20$, and so

$$[\mathbf{q}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -4 \\ 20 \end{pmatrix}.$$