MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 4 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW4 due date.

Exercise 2. (a) Theorem 4.38 from lectures says that *every* set of more than 3 vectors in $\mathbb{R}^3$ is linearly dependent, so in particular these 4 vectors are linearly dependent.

(b) Theorem 4.12 from lectures says that 3 vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are linearly independent if and only if the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has non-zero determinant. Since

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 5 & 2 & 7 \end{vmatrix} = 0,$$

as you should be able to check, we see that the given vectors are linearly dependent.

(c) The given vectors are linearly independent because

$$\begin{vmatrix} 3 & 3 & -6 \\ -2 & -1 & 4 \\ 1 & 4 & -1 \end{vmatrix} = 3 \neq 0.$$

Since there are 3 vectors and $\mathbb{R}^3$ has dimension 3, the vectors form a basis for $\mathbb{R}^3$

(d) A set of *two* vectors is linearly dependent if and only if one of the vectors is a scalar multiple of the other. Since this is clearly not the case for the two given vectors, they are linearly independent. However, they do not form a basis for $\mathbb{R}^3$, because $\mathbb{R}^3$ has dimension $3 > 2$.

Exercise 3. (a) Since $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V, every other vector $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$, and so by definition of linear independence, the set $\{\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ cannot be linearly independent.

(b) The set $S \setminus \{\mathbf{v}_1\} = \{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ does not span V because $\mathbf{v}_1$ cannot be written as a linear combination of $\mathbf{v}_2, \dots, \mathbf{v}_n$. (If $\mathbf{v}_1$ could be written as such a linear combination, then S would not have been linearly independent, contrary to our assumption.)

(c) Write $\mathbf{v}_i = A\mathbf{x}_i$ for each $i \in \{0, \dots, n\}$, and suppose that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

for some scalars $c_1, \dots, c_n$. We are trying to show that $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent, so we must show that $c_1, \dots, c_n$ are all equal to 0. We can re-write the above equation as

$$\mathbf{0} = c_1(A\mathbf{x}_1) + \cdots + c_n(A\mathbf{x}_n) = A(c_1\mathbf{x}_1 + \cdots c_n\mathbf{x}_n).$$

Since A is invertible, the system $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution, and so we may conclude that the vector

$$c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$$

must be $\mathbf{0}$. However, the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent (by assumption), so this implies that $c_1 = \cdots = c_n = 0$, which is what we were trying to prove.

Exercise 4. Note that this is Theorem 4.32 from the lecture notes. Suppose that we can write $\mathbf{v}$ as two linear combinations:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n, \tag{1}$$

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n. \tag{2}$$

Note that subtracting (2) from (1) gives

$$(\alpha_1 - \beta_1)\mathbf{v}_1 + \cdots + (\alpha_n - \beta_n)\mathbf{v}_n = \mathbf{0}. \tag{3}$$

Now suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. We must prove that the linear combinations (1) and (2) are actually the same. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, we know that the system (3) has *only* the trivial solution, i.e. $\alpha_1 - \beta_1 = \cdots = \alpha_n - \beta_n = 0$. Hence, $\alpha_i = \beta_i$ for each $i \in \{1, \ldots, n\}$, and so the two linear combinations are indeed the same.

Conversely, if the linear combinations (1) and (2) are actually different, then we must have $\alpha_i \neq \beta_i$ for *some* i, and hence (3) has a non-trivial solution (because the weight $\alpha_i - \beta_i$ of $\mathbf{v}_i$ is not 0), which means that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *not* linearly independent.

Exercise 5. (a) Using the hint, we can write every $\mathbf{p} \in H$ in the form

$$\begin{aligned} \mathbf{p}(t) &= (t-1)(at^2 + bt + c) \\ &= at^2(t-1) + bt(t-1) + c(t-1) \end{aligned}$$

for some scalars a, b, c. From this we immediately see that H is spanned by the polynomials $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ given by

$$\mathbf{p}_1(t) = t^2(t-1), \quad \mathbf{p}_2(t) = t(t-1), \quad \mathbf{p}_3(t) = t - 1.$$

If we can show that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent, then we will have found a basis for H and be able to conclude that H has dimension 3. Suppose therefore that

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$$

for some scalars c_1, c_2, c_3. We must prove that $c_1 = c_2 = c_3 = 0$. The above equation says that, for every $t \in \mathbb{R}$, we have

$$c_1t^2(t-1) + c_2t(t-1) + c_3(t-1) = 0,$$

or in other words,

$$c_1t^3 + (c_2 - c_1)t^2 + (c_3 - c_2)t - c_3 = 0.$$

Since the polynomial on the left-hand side above must evaluate to 0 for *every* $t \in \mathbb{R}$, it must be that case that all of its coefficients are equal to 0. This gives us a system of four equations for the three unknowns c_1, c_2, c_3, and you can easily check that this system has only the trivial solution $c_1 = c_2 = c_3 = 0$ (indeed, two of the equations say that $c_1 = 0$ and $c_3 = 0$, and then either of the remaining two equations gives $c_2 = 0$). Therefore, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are indeed linearly independent, and hence form a basis for H. In particular, $\dim(H) = 3$.

(b) By definition, the subspace H is the solution set of the linear system

$$\begin{aligned} r - 2s + t + 3u &= 0 \\ s + t - 4u &= 0. \end{aligned}$$

The free variables are t and u, and the leading variables are r and s. Writing $\alpha = t$ and $\beta = u$, we find that $s = -\alpha + 4\beta$ and then $r = 2(4\beta - \alpha) - \alpha - 3\beta = -3\alpha + 5\beta$. Therefore,

$$\begin{aligned} H &= \{(-3\alpha + 5\beta, -\alpha + 4\beta, \alpha, \beta)^T : \alpha, \beta \in \mathbb{R}\} \\ &= \{\alpha(-3, -1, 1, 0)^T + \beta(5, 4, 0, 1)^T : \alpha, \beta \in \mathbb{R}\} \\ &= \text{span}((-3, -1, 1, 0)^T, (5, 4, 0, 1)^T). \end{aligned}$$

Moreover, the vectors $(-3, -1, 1, 0)^T$ and $(5, 4, 0, 1)^T$ are linearly independent because they are not scalar multiples of each other, so in fact they form a basis for H. Hence, H has dimension 2.

(c) The set H of upper triangular matrices in $\mathbb{R}^{3\times 3}$ is spanned by the six matrices

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

because any upper triangular matrix

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

can be written as the linear combination

$$c_{11}E_{11} + c_{12}E_{12} + c_{13}E_{13} + c_{22}E_{22} + c_{23}E_{23} + c_{33}E_{33}.$$

Moreover, $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$ are linearly independent (and hence a basis for H), because if we attempt to set the above linear combination equal to the zero matrix then we must clearly have $c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}$ all equal to 0. In particular, $\dim(H) = 6$.