MTH5112 Linear Algebra I

COURSEWORK 4

WebWork submission of **exercise marked** $(*)$ due:
11.59am on Wednesday 01 November 2023

You should also attempt all of the other exercises in order develop your mathematical reasoning and skill in constructing arguments and proofs; model solutions will be posted on QMPlus after the submission date.

Exercise $(*)$ 1. Solve WeBWork **Set 4** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023/`.

Log in with your 'ah***' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. Determine which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent, and which of the linearly independent sets are bases for $\mathbb{R}^3$:

(a) $(1,1,1)^T, (3,4,3)^T, (2,1,3)^T, (1,1,3)^T$;
(b) $(2,-1,5)^T, (1,3,2)^T, (3,2,7)^T$;
(c) $(3,3,-6)^T, (-2,-1,4)^T, (1,4,-1)^T$;
(d) $(1,2,3)^T, (4,5,0)^T$.

Exercise 3. Prove the following statements.

(a) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set for a vector space V, then the set $S \cup \{\mathbf{v}\}$ is linearly dependent for every vector $\mathbf{v} \in V \setminus S$.
(b) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in a vector space V, then the set $S \setminus \{\mathbf{v}_1\}$ does not span V.
(c) If $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent vectors in $\mathbb{R}^n$, and A is an invertible $n \times n$ matrix, then the vectors $A\mathbf{x}_1, \ldots, A\mathbf{x}_k$ are also linearly independent.

Exercise 4. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V, and let $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Prove that $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in a *unique* way (that is, with uniquely determined weights/coefficients) if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Exercise 5. In each of the following cases, find a basis for the subspace H of the vector space V (and *prove* that it is a basis), and thereby determine the dimension of H.

(a) $V = P_3$ and $H = \{\mathbf{p} \in P_3 : \mathbf{p}(1) = 0\}$. (Hint: every polynomial in H can be written in the form $\mathbf{p}(t) = (t-1)(at^2 + bt + c)$ for some $a, b, c \in \mathbb{R}$.)
(b) $V = \mathbb{R}^4$ and $H = \{(r,s,t,u)^T : r - 2s + t + 3u = 0 \text{ and } s + t - 4u = 0\}$.
(c) $V = \mathbb{R}^{3\times 3}$ and H is the set of upper triangular matrices.