Summary of results about spanning sets,
linear independence
bases and dimension :

Assume :
- V v.sp., $\dim(V) = d$
- $S \subseteq V$ a spanning set
- $I \subseteq V$ a linearly independent set.

Then :
- $|I| \leq d \leq |S|$
- if $|I| = d$, then I is a basis
- if $|S| = d$, then S is a basis.

Moreover :
- I is contained in a basis
- S contains a basis.

Coordinates

Th Let V be a v.sp., $\underline{v}_1, \dots, \underline{v}_n \in V$.

Each vector $\underline{v} \in \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n)$ can be written <u>uniquely</u> as a linear combination of $\underline{v}_1, \dots, \underline{v}_n$ iff $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

Pf $\boxed{\Leftarrow}$ Supp. $\underline{v}_1, \dots, \underline{v}_n$ are lin. indep., $\underline{v} \in \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n)$.

Then $\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

Supp. also $\underline{v} = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$ — — $\beta_1 \dots \beta_n \in \mathbb{R}$.

$\Rightarrow$ $\underline{0} = \underline{v} - \underline{v} = (\alpha_1 - \beta_1)\underline{v}_1 + \dots + (\alpha_n - \beta_n)\underline{v}_n$

Because $\underline{v}_1, \dots, \underline{v}_n$ are lin. indep. $\Rightarrow$ $\alpha_1 - \beta_1 = 0, \dots, \alpha_n - \beta_n = 0$

i.e. $\alpha_1 = \beta_1, \dots, \alpha_n = \beta_n$

so the representation is unique.

$\boxed{\Rightarrow}$ Assume we have "unique representation"

Suppose $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$;

trivially also $0\,\underline{v}_1 + \dots + 0\,\underline{v}_n = \underline{0}$

so uniqueness forces $\alpha_1 = 0, \dots, \alpha_n = 0$, hence

$\underline{v}_1, \dots, \underline{v}_n$ are lin. indep. $\square$

Remark Supp. $B = (\underline{b}_1, \dots, \underline{b}_n)$ is a basis for a v.sp. V.

Since the basis is a lin. indep. spanning set, Theorem tells us that, for every $\underline{v} \in V$

there exist unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t.

$$\underline{v} = \alpha_1 \underline{b}_1 + \dots + \alpha_n \underline{b}_n .$$

Hence, $\underline{v} \in V$ is uniquely determined by its coordinate vector with respect to the basis B.

$$[\underline{v}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

The scalars $\alpha_1, \dots, \alpha_n$ are the coordinates of $\underline{v}$ in the basis B.

Ex ① Every vector $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ is equal to its own coordinate vector w.r.t. the standard basis $E = (\underline{e}_1, \dots, \underline{e}_n)$.

$$[\underline{v}]_E = \underline{v} .$$

② Let $E = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ be the standard basis for $\mathbb{R}^2$

$B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$ be another - - - - .

Let $\underline{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. Then $[\underline{v}]_E = \underline{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$, since $\underline{v} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

but $[\underline{v}]_B = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ since $\begin{pmatrix} 1 \\ 6 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

<u>Th</u> Let $\mathcal{B} = (\underline{b}_1, \dots, \underline{b}_n)$ be a basis for $\mathbb{R}^n$.

Let $P_{\mathcal{B}}$ be the $n \times n$ matrix whose j-th column is $\underline{b}_j$.

Then $P_{\mathcal{B}}$ is invertible, and, for every $\underline{x} \in \mathbb{R}^n$,

$$\underline{x} = P_{\mathcal{B}} [\underline{x}]_{\mathcal{B}} .$$

<u>Pf</u> If $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, then $\underline{x} = \alpha_1 \underline{b}_1 + \dots + \alpha_n \underline{b}_n$

so $$\underline{x} = \underbrace{\left(\underline{b}_1 \middle| \cdots \middle| \underline{b}_n \right)}_{P_{\mathcal{B}}} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = P_{\mathcal{B}} [\underline{x}]_{\mathcal{B}} .$$

Moreover, $P_{\mathcal{B}}$ is invertible because its columns are lin. indep. $\square$

We call $P_{\mathcal{B}}$ the <u>transition matrix from $\mathcal{B}$ to the standard basis</u>.

<u>Ex</u> Let $\mathcal{B} = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$, and $\underline{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \in \mathbb{R}^2$.

Find the $\mathcal{B}$-coordinates of $\underline{x}$, i.e., find $[\underline{x}]_{\mathcal{B}}$.

Since $\underline{x} = P_{\mathcal{B}} [\underline{x}]_{\mathcal{B}} \Rightarrow [\underline{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} \underline{x}$

We have $P_{\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, so we compute $P_{\mathcal{B}}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$.

$$\Rightarrow [\underline{x}]_{\mathcal{B}} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} .$$

Th Let B and D be bases for $\mathbb{R}^n$.

For any $\underline{x} \in \mathbb{R}^n$,

$$[\underline{x}]_B = P_B^{-1} P_D [\underline{x}]_D .$$

Pf By the previous Theorem,

$$P_B [\underline{x}]_B = \underline{x} = P_D [\underline{x}]_D$$

as P_B is invertible

$$\Rightarrow \quad [\underline{x}]_B = P_B^{-1} P_D [\underline{x}]_D \qquad \square$$

The $n \times n$ matrix $P_B^{-1} P_D$ is the <u>transition matrix from D to B</u>.

Ex $B = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$, $D = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$

If the D-coordinates of some $\underline{x} \in \mathbb{R}^2$ are $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$
what are the B-coordinates of $\underline{x}$?

$$[\underline{x}]_B = P_B^{-1} P_D [\underline{x}]_B = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} .$$

Why consider coordinatization?

- to "translate" problems about abstract v.sp. into ----- $\mathbb{R}^n$.

<u>Ex</u> Consider the standard basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ in $\mathbb{R}^{2\times 2}$

Given a matrix $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$, we can write

$$A = \alpha_{11} E_{11} + \alpha_{12} E_{12} + \alpha_{21} E_{21} + \alpha_{22} E_{22},$$ so

$$[A]_{\mathcal{B}} = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{pmatrix} \in \mathbb{R}^4$$

CW: Prove

$$[A+A']_{\mathcal{B}} = [A]_{\mathcal{B}} + [A']_{\mathcal{B}}$$

$$[\alpha A]_{\mathcal{B}} = \alpha [A]_{\mathcal{B}}.$$

Idea: the map $[-]_{\mathcal{B}} : \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ can "translate" problems about matrices into problems about column vectors.

<u>Ex</u> Let $\mathcal{B} = (1, t, t^2, t^3)$ be the std. basis for P_3

For a polynomial $p(t) = at^3 + bt^2 + ct + d$, we have

$$[p]_{\mathcal{B}} = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \in \mathbb{R}^4.$$

CW: Prove

$$[p+q]_{\mathcal{B}} = [p]_{\mathcal{B}} + [q]_{\mathcal{B}}$$

$$[\alpha p]_{\mathcal{B}} = \alpha [p]_{\mathcal{B}}.$$

The map $[-]_{\mathcal{B}} : P^3 \to \mathbb{R}^4$ can "translate" problems about polynomials into problems about column vectors.

Linear Transformations

<u>Def</u> Let V, W be vector spaces

A <u>linear transformation</u> (or <u>linear map/operator</u>)

is a function

$$L : V \to W \quad \text{satisfying}$$

(1) $L(\underline{u}+\underline{v}) = L(\underline{u}) + L(\underline{v})$ for all $\underline{u}, \underline{v} \in V$

(2) $L(\alpha \underline{u}) = \alpha L(\underline{u})$ for all $\alpha \in \mathbb{R}$, $\underline{u} \in V$

<u>Lemma</u> Let $L : V \to W$ be a linear transformation

Then:

(a) $L(\underline{0}_V) = \underline{0}_W$

(b) $L(-\underline{v}) = -L(\underline{v})$

(c) $L\left(\sum_{i=1}^{n} \alpha_i \underline{v}_i\right) = \sum_{i=1}^{n} \alpha_i L(\underline{v}_i)$,

<u>Pf</u> (a) $L(\underline{0}) = L(0 \cdot \underline{0}) \overset{(2)}{=} 0\, L(\underline{0}) = \underline{0}$.

(b) $L(-\underline{v}) = L((-1)\underline{v}) \overset{(2)}{=} (-1)\, L(\underline{v}) = -L(\underline{v})$.

(c) $L\left(\sum_{i=1}^{n} \alpha_i \underline{v}_i\right) = \sum_{i=1}^{n} L(\alpha_i \underline{v}_i) = \sum_{i=1}^{n} \alpha_i L(\underline{v}_i)$.

apply (1) $(n-1)$ times

apply (2) to each term

$\square$

<u>Ex</u> The identity map

$$Id : V \to V, \text{ defined by}$$

$$Id(\underline{v}) = \underline{v}, \text{ for } \underline{v} \in V$$

is a linear transformation, because

(1) $Id(\underline{u} + \underline{v}) = \underline{u} + \underline{v} = Id(\underline{u}) + Id(\underline{v})$

(2) $Id(\alpha \underline{u}) = \alpha \underline{u} = \alpha\, Id(\underline{u})$

<u>Ex</u> The map $L : V \to W$

$$L(\underline{v}) = \underline{0} \quad \text{for } \underline{v} \in V$$

is linear [CW].

<u>Examples of linear maps</u>

<u>Ex</u> $L : \mathbb{R}^2 \to \mathbb{R}^2$ def. by

$$L(\underline{x}) = 2\underline{x} \quad \text{for} \quad \underline{x} \in \mathbb{R}^2.$$

L is linear, because for all $\underline{x}, \underline{y} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, we have

(1) $L(\underline{x} + \underline{y}) = 2(\underline{x} + \underline{y}) = 2\underline{x} + 2\underline{y} = L(\underline{x}) + L(\underline{y})$

(2) $L(\alpha \underline{x}) = 2(\alpha \underline{x}) = (2\alpha)\underline{\alpha} = \alpha(2\underline{x}) = \alpha L(\underline{x})$.

Ex (cw) More generally, fix $\lambda \in \mathbb{R}$, and let V v.sp.

The operator

$$L : V \to V$$

$$L(\underline{v}) = \lambda \underline{v}, \quad \text{for } \underline{v} \in V$$

is linear.

Ex $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ is linear;

for all $\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{=\underline{x}}, \underbrace{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}_{=\underline{y}} \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$, we have

(1) $$L(\underline{x}+\underline{y}) = L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = L\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}\right) = \begin{pmatrix} -(x_2+y_2) \\ x_1+y_1 \end{pmatrix}$$

$$= \begin{pmatrix} -x_2 + (-y_2) \\ x_1 + y_1 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} = L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + L\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$

$$= L(\underline{x}) + L(\underline{y}).$$ ✓

(2) $$L(\alpha \underline{x}) = L\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = L\left(\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}\right) = \begin{pmatrix} -\alpha x_2 \\ \alpha \cdot x_1 \end{pmatrix} =$$

$$= \alpha \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \alpha\, L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \alpha\, L(\underline{x}).$$ ✓

Ex Projection maps

$p_1 : \mathbb{R}^2 \to \mathbb{R}$, $\quad p_2 : \mathbb{R}^2 \to \mathbb{R}$ are linear.

$p_1\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 \quad p_2\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_2 \quad$ (cw).

__Ex__ $L : \mathbb{R}^2 \to \mathbb{R}^1 = \mathbb{R}$

$$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \sqrt{x_1^2 + x_2^2} \quad \text{is NOT linear.}$$

For example $L\left(-\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = L\left(\begin{pmatrix} -3 \\ -4 \end{pmatrix}\right) = \sqrt{9+16} = 5$

$$\neq$$

$$-L\left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = -\sqrt{9+16} = -5$$

so L fails property (2) from defn.

__Remark__ A matrix $A \in \mathbb{R}^{m \times n}$ induces a linear transformation

$$\mathcal{L}_A : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathcal{L}_A(\underline{x}) = A\underline{x}, \quad \text{for } \underline{x} \in \mathbb{R}^n.$$

Indeed, for $\underline{x}, \underline{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(1) $\mathcal{L}_A(\underline{x} + \underline{y}) = A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y} = \mathcal{L}_A(\underline{x}) + \mathcal{L}_A(\underline{y})$.

(2) $\mathcal{L}_A(\alpha \underline{x}) = A(\alpha \underline{x}) = \alpha A\underline{x} = \alpha \mathcal{L}_A(\underline{x})$.

Th If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a matrix $A \in \mathbb{R}^{m \times n}$ s.t.

$$L = \mathcal{L}_A \,, \quad \text{i.e.,}$$

$$L(\underline{x}) = A\underline{x} \quad \text{for all } \underline{x} \in \mathbb{R}^n .$$

Pf Let $(\underline{e}_1, \dots, \underline{e}_n)$ be the standard basis for $\mathbb{R}^n$.

Let
$$A = \left(L(\underline{e}_1) \middle| \cdots \middle| L(\underline{e}_n) \right) \in \mathbb{R}^{m \times n}$$
be the matrix with columns $L(\underline{e}_1), \dots, L(\underline{e}_n)$.

Then
$$L(\underline{x}) = L\left(\sum_{i=1}^n x_i \underline{e}_i \right) \overset{\overset{L \text{ linear}}{\downarrow}}{=} \sum_{i=1}^n x_i L(\underline{e}_i) \underset{\text{defn. of matrix mult.}}{=} A\underline{x}$$
$\square$

Ex Consider $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

Find the associated matrix.

$$L(\underline{e}_1) = L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;; \quad L(\underline{e}_2) = L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

The corr. matrix is

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{CW: Check } A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Ex Let B be a basis for an n-dimensional v.sp. V

The coordinatisation map

$$[-]_B : V \longrightarrow \mathbb{R}^n \quad \text{is linear,}$$

$$[\underline{u}+\underline{v}]_B = [\underline{u}]_B + [\underline{v}]_B$$

$$[\alpha \underline{u}]_B = \alpha [\underline{u}]_B \, .$$

Examples of linear transformations from Calculus / Analysis

(i) $$L : C[a,b] \to \mathbb{R}^1 \simeq \mathbb{R}$$

$$L(f) = \int_a^b f(x)\, dx \quad \text{is a linear transformation,}$$

because, for all $f, g \in C[a,b]$, all $\alpha \in \mathbb{R}$, Calculus

$$(1) \quad L(f+g) = \int_a^b (f+g)(x)\, dx = \int_a^b f(x) + g(x)\, dx =$$

$$= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx = L(f) + L(g).$$

Calculus

$$(2) \quad L(\alpha f) = \int_a^b (\alpha f)(x)\, dx = \int_a^b \alpha f(x)\, dx = \alpha \int_a^b f(x)\, dx = \alpha L(f).$$

Ex Let $C^1[a,b]$ denote the subspace of $C[a,b]$ of differentiable functions whose derivatives are continuous.

Define
$$D : C^1[a,b] \to C[a,b]$$
$$D(f) = f' \quad (\text{the derivative of } f).$$

Then D is a linear transformation.

For all $f, g \in C^1[a,b]$, all $\alpha \in \mathbb{R}$,

Calculus

(1) $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$.

(2) $D(\alpha f) = (\alpha f)' = \alpha f' = \alpha D(f)$.

Ex If P_n denotes polynomials of degree $\leq n$, we have a derivation operator

$$D : P_n \longrightarrow P_{n-1}$$
$$D(p) = p', \quad \text{i.e.,} \quad D\left(\sum_{i=0}^{n} \alpha_i t^i\right) = \sum_{i=1}^{n} i \alpha_i t^{i-1}$$

and an integration operator

$$I : P_n \to P_{n+1}$$
$$I(p) = \int_0^t p(x)\,dx, \quad \text{i.e.} \quad I\left(\sum_{i=0}^{n} \alpha_i t^i\right) = \sum_{i=0}^{n} \frac{\alpha_i}{i+1} t^{i+1}$$

which are both linear.

<u>Ex</u> $S : C(\mathbb{R}) \to C(\mathbb{R})$ the "shift" operator

$$[S(f)](x) = f(x+1), \quad x \in \mathbb{R}$$

S is linear.

<u>Ex</u> Let $T : C[a,b] \to C[a,b]$

$$[T(f)](x) = f(x) + 1 \quad \text{for } x \in [a,b].$$

Then T is <u>not linear</u> because $T(\underline{0}) \neq \underline{0}$.

Forming new linear transformations

Def Let $L, L' : V \to W$, $M : W \to Z$ be linear maps, $\alpha \in \mathbb{R}$.

Define:

(i) $(L+L') : V \to W$, $(L+L')(\underline{v}) = L(\underline{v}) + L'(\underline{v})$, for $\underline{v} \in V$.

(ii) $\alpha L : V \to W$, $(\alpha L)(\underline{v}) = \alpha\, L(\underline{v})$, for $\underline{v} \in V$.

(iii) $M \circ L : V \to Z$, $(M \circ L)(\underline{v}) = M(L(\underline{v}))$, for $\underline{v} \in V$.

composite

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[rr, bend right, "M \circ L"'] & W \arrow[r, "M"] & Z \end{tikzcd}$$

Lemma $L+L'$, αL, $M \circ L$ are linear maps.

Pf • $L+L'$ is linear:

L, L' are linear

(1) $(L+L')(\underline{u}+\underline{v}) = L(\underline{u}+\underline{v}) + L'(\underline{u}+\underline{v}) =$

$= L(\underline{u}) + L(\underline{v}) + L'(\underline{u}) + L'(\underline{v}) = \big(L(\underline{u}) + L'(\underline{u})\big) + \big(L(\underline{v}) + L'(\underline{v})\big)$

$= (L+L')(\underline{u}) + (L+L')(\underline{v})$.

L, L' are linear

(2) $(L+L')(\alpha\underline{u}) = L(\alpha\underline{u}) + L'(\alpha\underline{u}) =$

$= \alpha\, L(\underline{u}) + \alpha\, L'(\underline{u}) = \alpha\big(L(\underline{u}) + L'(\underline{u})\big) = \alpha(L+L')(\underline{u})$

• αL is linear: CW

- $M \circ L$ is linear:

(1) $$(M\circ L)(\underline{u}+\underline{v}) = M\big(L(\underline{u}+\underline{v})\big) \overset{L \text{ linear}}{=} M\big(L(\underline{u})+L(\underline{v})\big) \overset{M \text{ linear}}{=}$$

$$= M(L(\underline{u})) + M(L(\underline{v})) = (M\circ L)(\underline{u}) + (M\circ L)(\underline{v})$$

(2) $$(M\circ L)(\alpha\underline{u}) = M(L(\alpha \underline{u})) \overset{L \text{ linear}}{=} M(\alpha L(\underline{u})) \overset{M \text{ linear}}{=}$$

$$= \alpha M(L(\underline{u})) = \alpha (M\circ L)(\underline{u}).$$

$\square$

<u>Exercise</u>: Recall, given a matrix $A \in \mathbb{R}^{m\times n}$, we have the associated linear map $\mathcal{L}_A : \mathbb{R}^n \to \mathbb{R}^m$

$$\mathcal{L}_A(\underline{x}) = A\underline{x}.$$

Prove, if $A, A' \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $\alpha \in \mathbb{R}$, then

(i) $\mathcal{L}_A + \mathcal{L}_{A'} = \mathcal{L}_{A+A'}$

(ii) $\alpha \mathcal{L}_A = \mathcal{L}_{\alpha A}$

(iii) $\mathcal{L}_A \circ \mathcal{L}_B = \mathcal{L}_{AB}$

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "\mathcal{L}_B"] \arrow[rr, bend right, "\mathcal{L}_{AB} = \mathcal{L}_A \circ \mathcal{L}_B"'] & \mathbb{R}^n \arrow[r, "\mathcal{L}_A"] & \mathbb{R}^m \end{tikzcd}$$