MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 3 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW3 due date.

Exercise 2. Each of the sets S_1, S_2, S_3 contains four vectors in $\mathbb{R}^4$, call them $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$. These vectors span $\mathbb{R}^4$ if, given *any* vector $\mathbf{w} = (a,b,c,d)^T \in \mathbb{R}^4$, we can find scalars α_1, α_2, α_3, α_4 such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 = \mathbf{w}.$$

This is just a linear system in the unknowns α_1, α_2, α_3, α_4, and so $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ span $\mathbb{R}^4$ precisely if this system has a solution *for every* $\mathbf{w} \in \mathbf{R}^4$. Therefore, we can just form an augmented matrix (with *columns* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{w}$) and attempt to solve the system in the usual way.

(a) The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 2 & -1 & 0 & 2 & b \\ 0 & 1 & 3 & -3 & c \\ 0 & 0 & 1 & -1 & d \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 0 & -1 & 0 & 0 & b-2a \\ 0 & 1 & 3 & -3 & c \\ 0 & 0 & 1 & -1 & d \end{array}\right) R_2 \to R_2 - 2R_1$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 0 & -1 & 0 & 0 & b-2a \\ 0 & 0 & 3 & -3 & c+b-2a \\ 0 & 0 & 1 & -1 & d \end{array}\right) R_3 \to R_3 + R_2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 0 & -1 & 0 & 0 & b-2a \\ 0 & 0 & 3 & -3 & c+b-2a \\ 0 & 0 & 0 & 0 & d-\frac{1}{3}(c+b-2a) \end{array}\right) R_4 \to R_4 - \tfrac{1}{3}R_3 \quad .$$

From the final row we see that $d - \frac{1}{3}(c+b-2a)$ must be equal to 0. Therefore, the vectors in the set S_1 do *not* span $\mathbb{R}^4$, because in order for the above system to have a solution, the vector $\mathbf{w} = (a,b,c,d)^T$ must at least be chosen in such a way that $d - \frac{1}{3}(c+b-2a) = 0$, i.e. the system will not have a solution for *every* choice of $\mathbf{w}$.

(b) The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & a \\ 0 & 1 & 0 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right),$$

from which we immediately see that the system has a unique solution for each choice of $\mathbf{w} = (a,b,c,d)^T$: the fourth row says that $\alpha_4 = d$; then the third row says that $\alpha_3 + \alpha_4 = c$, i.e. $\alpha_3 = c - \alpha_4 = c - d$; then similarly the second row says that $\alpha_2 = b - \alpha_4 = b - d$; and finally the first row says that $\alpha_1 = a - \alpha_4 = a - d$. In other words, we have

$$\begin{aligned}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} &= \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &= (a-d) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (b-d) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (c-d) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.\end{aligned}$$

Therefore, the set S_2 spans $\mathbb{R}^4$.

(c) The augmented matrix is

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 1 & a \\ 0 & 1 & 1 & 1 & b \\ 1 & 0 & -1 & 1 & c \\ 0 & 0 & 0 & 0 & d \end{array}\right),$$

and we immediately see that the last row forces d to be equal to 0. Therefore, the vectors in the given set S_3 do *not* span $\mathbb{R}^4$, because the above system can only have a solution if the vector $\mathbf{w} = (a, b, c, d)^T$ is chosen with $d = 0$.

Exercise 3. We must show that

$$\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

for every choice of vector $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, and we will do this by proving that each of these two sets is a subset of the other.

To show that span$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a subset of span$(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$, we must take an arbitrary vector $\mathbf{u}$ in span$(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$ and prove that $\mathbf{u}$ is also contained in span$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. This is easy: since $\mathbf{u} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, the definition of "span" says that there exist scalars $c_1, \ldots, c_n$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n,$$

but we can just re-write this equation as

$$\mathbf{u} = (c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n) + 0\mathbf{v},$$

from which we see that $\mathbf{u} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$.

To show that span$(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$ is a subset of span$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, we must take an arbitrary vector $\mathbf{w}$ in span$(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v})$ and prove that $\mathbf{w}$ is also contained in span$(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. We know that

$$\mathbf{w} = \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n + \gamma\mathbf{v}$$

for some scalars $\beta_1, \ldots, \beta_n, \gamma$. On the other hand, since $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, there are scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots \alpha_n\mathbf{v}_n,$$

and so in fact we can write

$$\begin{aligned}\mathbf{w} &= \beta_1\mathbf{v}_1 + \cdots + \beta_n\mathbf{v}_n + \gamma(\alpha_1\mathbf{v}_1 + \cdots \alpha_n\mathbf{v}_n) \\ &= (\beta_1 + \gamma\alpha_1)\mathbf{v}_1 + \cdots + (\beta_n + \gamma\alpha_1)\mathbf{v}_n.\end{aligned}$$

That is, we have found a way to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, so we have $\mathbf{w} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Exercise 4. (a) Every diagonal 2×2 matrix has $(1,2)$ and $(2,1)$ entries equal to 0, so every linear combination of diagonal matrices also has $(1,2)$ and $(2,1)$ entries equal to 0. Hence, for example, the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

cannot be written as a linear combination of diagonal matrices because its $(2,1)$ entry is not 0, and so the set of diagonal matrices does not span $\mathbb{R}^{2\times 2}$.

(b) Every upper triangular 2×2 matrix has $(2,1)$ entry equal to 0, so every linear combination of upper triangular matrices also has $(2,1)$ entry equal to 0. In particular, the matrix A in part (a) cannot be written as a linear combination of upper triangular matrices because its $(2,1)$ entry is not 0, so the set of upper triangular matrices does not span $\mathbb{R}^{2\times 2}$.

(c) Every symmetric 2×2 matrix has $(1,2)$ and $(2,1)$ entries that are equal to each other, so every linear combination of symmetric matrices also has this property. Hence, again, the matrix A in part (a) cannot be written as a linear combination of symmetric matrices, because its $(1,2)$ and $(2,1)$ entries are not equal (the former is 0 while the latter is 1). Therefore, the set of symmetric matrices does not span $\mathbb{R}^{2\times 2}$.

Exercise 5. (a) If $\mathbf{p}$ is any linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$, then we have

$$\mathbf{p}(x) = \alpha_1 \mathbf{p}_1(x) + \alpha_2 \mathbf{p}_2(x) = \alpha_1 x^2 + \alpha_2 x + (\alpha_1 - \alpha_2)$$

for some scalars α_1, α_2. Therefore, a polynomial $ax^2 + bx + c$ can only be written as a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ if the coefficient c is equal to $a - b$. Hence, for example, the polynomial $2x^2 + x$ *cannot* be written a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$, so $S_1 = \{\mathbf{p}_1, \mathbf{p}_2\}$ does *not* span P_2.

(b) If we want to write a polynomial $\mathbf{q}$ with $\mathbf{q}(x) = ax^2 + bx + c$ as a linear combination of the given polynomials $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ then we must find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$ax^2 + bx + c = \alpha_1(2x^2 - 1) + \alpha_2(x + 1) + \alpha_3(x + 2)$$

for all $x \in \mathbb{R}$. Rearranging the right-hand side, this means that

$$ax^2 + bx + c = (2\alpha_1)x^2 + (\alpha_2 + \alpha_3) + (\alpha_2 + 2\alpha_3 - \alpha_1)$$

for all $x \in \mathbb{R}$. Therefore, the coefficients of the various powers of x must agree on both sides of the equation, and so we obtain a linear system for the (unknown) scalars $\alpha_1, \alpha_2, \alpha_3$:

$$\begin{aligned} 2\alpha_1 \qquad\qquad &= a \\ \alpha_2 + \ \alpha_3 &= b \\ -\alpha_1 + \alpha_2 + 2\alpha_3 &= c. \end{aligned}$$

This system has a unique solution for every choice of $(a, b, c)^T$, given by

$$\alpha_1 = \frac{a}{2}, \quad \alpha_2 = -\frac{a}{2} + 2b - c, \quad \alpha_3 = \frac{a}{2} - b + c.$$

In other words, we can always write $\mathbf{q}$ as a linear combination of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ as follows:

$$ax^2 + bx + c = \frac{a}{2}(2x^2 - 1) + \left(-\frac{a}{2} + 2b - c\right)(x + 1) + \left(\frac{a}{2} - b + c\right)(x + 2),$$

and so S_2 does span P_2.

(c) If $\mathbf{r}$ is any linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ then

$$\mathbf{r}(x) = \alpha_1(x^2 + 2) + \alpha_2(x^2 + 5) + \alpha_3 = (\alpha_1 + \alpha_2)x^2 + (2\alpha_1 + 5\alpha_2 + \alpha_3)$$

for some scalars $\alpha_1, \alpha_2, \alpha_3$. Therefore, a polynomial $ax^2 + bx + c$ can only be written as a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ if the coefficient b of x is equal to 0. Hence, for example, the polynomial x cannot be written as a linear combination of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, and so these polynomials do not span P_2.

Exercise 6. (a) This is very similar to an example from lectures. Essentially, H is a subspace because the condition that defines whether $(x, y, z)^T$ is an element of H is a homogeneous linear equation in x, y and z. Let's prove this properly, though. H contains the zero vector $(0, 0, 0)^T$ because the coordinates of this vector certainly satisfy the given equation, i.e. $0+0+0 = 0$; in particular, H is non-empty. To prove closure of H under addition, we must take two arbitrary vectors $\mathbf{v} = (x, y, z)^T$ and $\mathbf{w}(x', y', z')^T$ in H, and show that their sum $\mathbf{v}+\mathbf{w} = (x+x', y+y', z+z')^T$ is also an element of H, i.e. that $(x+x')+(y+y')+(z+z') = 0$. Since $\mathbf{v}$ and $\mathbf{w}$ are in H, we know that $x + y + z = 0$ and $x' + y' + z' = 0$, so we get what we want pretty easily:

$$(x + x') + (y + y') + (z + z') = (x + y + z) + (z' + y' + z') = 0 + 0 = 0.$$

Therefore, H is closed under addition. To show that H is closed under scalar multiplication, we must take an arbitrary vector $\mathbf{v} = (x, y, z)^T$ in H and an arbitrary scalar α, and show that the vector $\alpha\mathbf{v} = (\alpha x, \alpha y, \alpha x)^T$ is an element of H, i.e. that its components satisfy the equation $(\alpha x) + (\alpha y) + (\alpha z) = 0$. Again, this is easy once we realise that we can use the fact that $x + y + z = 0$ (based on the assumption that $\mathbf{v} \in H$):

$$(\alpha x) + (\alpha y) + (\alpha z) = \alpha(x + y + z) = \alpha \cdot 0 = 0.$$

Therefore, H is closed scalar multiplication. This completes the proof that H is a subspace.

Now let's write down a spanning set for H. By definition, the vectors in H have the form $(x, y, -x - y)^T$, because they must satisfy the equation $x + y + z = 0$, which says that $z = -x - y$. However, every vector of the form $(x, y, -x - y)^T$ can be written as

$$(x, y, -x - y)^T = x(1, 0, -1)^T + y(0, 1, -1)^T,$$

i.e. a linear combination of the two vectors $(1, 0, -1)^T$ and $(0, 1, -1)^T$ with weights/coefficients x and y, respectively. Therefore, $\{(1, 0, -1)^T, (0, 1, -1)^T\}$ is a spanning set for H.

(b) The 2×2 zero matrix is certainly symmetric, so H is non-empty. We stated in lectures that the sum of any two symmetric matrices is symmetric (Proposition 2.16), and you were asked to prove this as an exercise, so we already know that H is closed under addition. It is also immediate from the definition of "scalar multiplication" of matrices that H is closed under scalar multiplication, because if a matrix $A = (a_{ij})_{2\times 2}$ is symmetric then $a_{12} = a_{21}$, and so the $(1, 2)$ and $(2, 1)$ entries αa_{12} and αa_{21} of any scalar multiple αA of A are also equal to each other. Hence, H is indeed a subspace of $\mathbb{R}^{2\times 2}$.

Let's now write down a spanning set for H. Every symmetric 2×2 matrix has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$. However, we can re-write this as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. as a linear combination of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

with weights a, b, c respectively. Therefore, these three matrices span H.

(c) Note that the polynomials in H are simply those whose x coefficient equals 0. H contains the zero polynomial (and so in particular is non-empty) because the zero polynomial can be written in the required form $ax^2 + c$ if we choose $a = c = 0$. H is closed under addition because if we take any $\mathbf{p}, \mathbf{q} \in H$, say $\mathbf{p}(x) = ax^2 + c$ and $\mathbf{q}(x) = a'x^2 + c'$, then $\mathbf{p} + \mathbf{q}$ is also in H because

$$(\mathbf{p} + \mathbf{q})(x) = (ax^2 + c) + (a'x^2 + c') = (a + a')x^2 + (c + c'),$$

which is the required form for a polynomial to be in H. Similarly, H is closed under scalar multiplication because for any scalar α we have

$$(\alpha\mathbf{p})(x) = (\alpha a)x^2 + (\alpha c).$$

Therefore, H is a subspace of P_2.

It remains to write down a spanning set for H that contains two vectors. Consider the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$ given by $\mathbf{p}_1(x) = x^2$ and $\mathbf{p}_2(x) = 1$. Both are elements of H because both have the form $ax^2 + c$ for some a and c (take $a = 1$ and $c = 0$ for $\mathbf{p}_1$, and take $a = 0$ and $c = 1$ for $\mathbf{p}_2$). Moreover, an *arbitrary* polynomial $ax^2 + c$ in H can be expressed as a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ because

$$ax^2 + c = a\mathbf{p}_1(x) + c\mathbf{p}_2(x),$$

i.e. the weights of the linear combination are a and c, respectively. Therefore, $H = \text{span}(\mathbf{p}_1, \mathbf{p}_2)$.