MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 3

WebWork submission of **exercise marked** $(*)$ due:
11.59am on Wednesday 01 November 2023

You should also attempt all of the other exercises in order develop your mathematical reasoning and skill in constructing arguments and proofs; model solutions will be posted on QMPlus after the submission date.

Exercise $(*)$ 1. Solve WeBWork **Set 3** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023/`.

Log in with your '`ah***`' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. Determine which of the following sets of vectors span $\mathbb{R}^4$.

(a) $S_1 = \{(1,2,0,0)^T, (0,-1,1,0)^T, (0,0,3,1)^T, (1,2,-3,-1)^T\}$.
(b) $S_2 = \{(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,0)^T, (1,1,1,1)^T\}$.
(c) $S_3 = \{(-1,0,1,0)^T, (2,1,0,0)^T, (0,1,-1,0)^T, (1,1,1,0)^T\}$.

Exercise 3. Let V be a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Prove that

$$\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

for every $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Exercise 4. Prove that *none* of the following sets of matrices span $\mathbb{R}^{2\times 2}$.

(a) The diagonal matrices.
(b) The upper triangular matrices.
(c) The symmetric matrices.

Exercise 5. Determine which of the following sets of polynomials span P_2.

(a) $S_1 = \{\mathbf{p}_1, \mathbf{p}_2\}$ where $\mathbf{p}_1(x) = x^2 + 1$ and $\mathbf{p}_2(x) = x - 1$.
(b) $S_2 = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ where $\mathbf{q}_1(x) = 2x^2 - 1$, $\mathbf{q}_2(x) = x + 1$ and $\mathbf{q}_3(x) = x + 2$.
(c) $S_3 = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ where $\mathbf{r}_1(x) = x^2 + 2$, $\mathbf{r}_2(x) = x^2 + 5$ and $\mathbf{r}_3(x) = 1$.

Exercise 6. In each of the following cases, prove that H is a subspace of V, and write down a set of n vectors that spans H (justifying your choice of spanning set).

(a) $V = \mathbb{R}^3$, $H = \{(x,y,z)^T : x + y + z = 0\}$, and $n = 2$.
(b) $V = \mathbb{R}^{2\times 2}$, H is the set of all symmetric 2×2 matrices, and $n = 3$.
(c) $V = P_2$, $H = \{\mathbf{p} \in P_2 : \mathbf{p}(x) = ax^2 + c \text{ for some } a, c \in \mathbb{R}\}$, and $n = 2$.

Exercise (MATLAB) 7. In this exercise you are encouraged to use the mathematical software MATLAB. This exercise provides you with some of the necessary code/commands needed to complete it, but you will also need to figure some things out for yourself by, e.g. consulting the help/documentation, finding similar examples online, figuring things out with your colleagues, and so on.

Recall *Cramer's rule* from Vectors and Matrices module: if A is an invertible $n \times n$ matrix, then the unique solution of any linear system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)} \quad \text{for } i \in \{1, \ldots, n\},$$

where

$$A_i(\mathbf{b})$$

is the $n \times n$ matrix obtained by replacing the ith column of A by the column vector $\mathbf{b}$.

MATLAB has an in-built command for replacing a column of a matrix by a given column vector. Specifically, if you wish to replace the ith column of a matrix `A` by a column vector `b` (of the appropriate size), you can type

```
A(:,[i])=b
```

Use this to verify Cramer's rule for the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 0 & 1 & 6 \\ 3 & -3 & 9 \\ 2 & 2 & 18 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 8 \end{pmatrix}.$$