Linear Independence

Def Let V v.sp.

The vectors $\underline{v}_1, \ldots, \underline{v}_n \in V$ are <u>linearly dependent</u> if there exist scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ <u>not all zero</u> s.t.

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}$$

Prop TFAE :

(i) $\underline{v}_1, \ldots, \underline{v}_n$ are lin. dependent ;

(ii) there exists a non-trivial solution to the linear system
$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}$$
in unknowns $\alpha_1, \ldots, \alpha_n$.

(iii) a non-trivial linear combination of $\underline{v}_1, \ldots, \underline{v}_n$ gives $\underline{0}$.

(iv) one of the vectors $\underline{v}_1, \ldots, \underline{v}_n$ can be written as a linear combination of the others ;

(v) for some $i \in \{1 \ldots n\}$, $\underline{v}_i \in \mathrm{Span}(\underline{v}_1, \ldots, \underline{v}_{i-1}, \underline{v}_{i+1}, \ldots \underline{v}_n)$.

<u>Pf</u> (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v) are just restatements.

<u>(iii) $\Rightarrow$ (iv)</u> Suppose $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$ and, say, $\alpha_i \neq 0$, some

Then $\alpha_i \underline{v}_i = - \sum_{j \neq i} \alpha_j \underline{v}_j \quad / \cdot \left(\frac{1}{\alpha_i}\right)$

$$\underline{v}_i = \left(\frac{-\alpha_1}{\alpha_i}\right) \underline{v}_1 + \dots + \left(-\frac{\alpha_{i-1}}{\alpha_i}\right) \underline{v}_{i-1} + \left(\frac{-\alpha_{i+1}}{\alpha_i}\right) \underline{v}_{i+1} + \dots + \left(\frac{-\alpha_n}{\alpha_i}\right) \underline{v}_n$$

$$\Rightarrow \quad \underline{v}_i \in \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_{i-1}, \underline{v}_{i+1}, \dots, \underline{v}_n)$$

<u>(v) $\Rightarrow$ (i)</u> Supp. $\underline{v}_i \in \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_{i-1}, \underline{v}_{i+1}, \dots, \underline{v}_n)$

$$\Rightarrow \quad \underline{v}_i = \beta_1 \underline{v}_1 + \dots + \beta_{i-1} \underline{v}_{i-1} + \beta_{i+1} \underline{v}_{i+1} + \dots + \beta_n \underline{v}_n$$

$$\Rightarrow \quad \beta_1 \underline{v}_1 + \dots + \beta_{i-1} \underline{v}_{i-1} + (-1) \underline{v}_i + \beta_{i+1} \underline{v}_{i+1} + \dots + \beta_n \underline{v}_n = \underline{0}$$

and this is a non-trivial lin. comb. since $(-1) \neq 0$. $\square$

$$\begin{tikzcd} & (i) \arrow[dl, Rightarrow] & \\ (ii) \arrow[d, Rightarrow] & & (v) \arrow[ul, Rightarrow] \\ (iii) \arrow[rr, Rightarrow] & & (iv) \arrow[u, Rightarrow] \end{tikzcd}$$

<u>Ex</u> Vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}$ are linearly dependent,

because $\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$ [one of them is a lin. comb. of the others]

i.e. $1\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \underline{0}$ [a non-trivial lin. comb. gives $\underline{0}$]

Def Let V v.sp., $S \subseteq V$ a subset.

Then S is lin. dependent if there exists $\{\underline{v}_1, \dots, \underline{v}_n\} \subseteq S$ s.t. $\underline{v}_1, \dots, \underline{v}_n$ are lin. dependent.

Remark Let V v.sp, $S \subseteq T \subseteq V$ subsets.

If S is lin. dependent $\Rightarrow$ T is lin. dependent.

[redacted]

Def Let V v.sp.

The vectors $\underline{v}_1, \dots, \underline{v}_n \in V$ are <u>linearly independent</u>

if $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$ forces all scalars $\alpha_1, \dots, \alpha_n$ to be 0.

[i.e., if they are <u>not</u> lin. dependent]

Prop TFAE:

(i) $\underline{v}_1, \dots, \underline{v}_n$ are lin. independent

(ii) $\forall \alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0} \Rightarrow \alpha_1 = \dots = \alpha_n = 0$.

(iii) the linear system

$$\alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n = \underline{0}$$

in unknowns $\alpha_1, \dots, \alpha_n$ only has the trivial solution $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \underline{0}$

(iv) only the trivial linear comb. of $\underline{v}_1 \dots \underline{v}_n$ gives $\underline{0}$.

(v) none of the vectors $\underline{v}_1, \dots, \underline{v}_n$ is a lin. comb. of the others.

(vi) for all $i \in \{1, \dots, n\}$, $\underline{v}_i \notin \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_{i-1}, \underline{v}_{i+1}, \dots, \underline{v}_n)$.

Pf All statements are logical negations of statements from the previous proposition. $\square$

<u>Ex</u> $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \in \mathbb{R}^2$ are lin. independent

Supp. scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy

$$\alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

$$\Rightarrow \quad \left.\begin{array}{r} \alpha_1 + 2\alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \end{array}\right| \overset{(\text{cw})}{\Longrightarrow} \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \text{ is the } \underline{\text{ONLY}} \text{ solution}$$

$$\Rightarrow \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \text{ are lin. indep.}$$

<u>Ex</u> $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}$ are lin. dependent

Need to find a non-trivial solution $\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$ to

$$\alpha_1\begin{pmatrix}1\\2\end{pmatrix} + \alpha_2\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Take, say, $\alpha_1 = -2$, $\alpha_2 = 1$.

<u>Def</u> V v.sp., $S \subseteq V$ subset.
We say that S is <u>linearly independent</u> if for every finite subset $\{\underline{v}_1, \dots, \underline{v}_n\} \subseteq S$, vectors $\underline{v}_1, \dots, \underline{v}_n$ are lin. indep.

<u>Remark</u> V v.sp., $S \subseteq T \subseteq V$ subsets.
If T is lin. indep $\Rightarrow$ S is lin. indep.

Ex The set $\{1, t, t^2, \dots\}$ is lin. indep. in the space P of all polynomials, because for all n, the polynomials $1, t, \dots, t^n$ are lin. indep:

$$\alpha_0 \cdot 1 + \alpha_1 t + \dots + \alpha_n t^n = \underline{0} \Rightarrow \alpha_0 = \alpha_1 = \dots = \alpha_n = 0.$$

Linear Independence Test

Let $\underline{a}_1, \dots, \underline{a}_n \in \mathbb{R}^m$. Decide whether they are lin. indep.

- Consider $A \in \mathbb{R}^{m \times n}$ with columns $\underline{a}_1, \dots, \underline{a}_n$.
- Write augmented matrix $\left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$ corresp. to system $A\underline{x} = \underline{0}$
- Bring $\left(\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right)$ to REF:
  - if no free variables (all vars. leading) $\rightsquigarrow$ INDEPENDENT
  - if have free variables $\rightsquigarrow$ DEPENDENT

Ex Are $\begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -4 \\ 14 \\ -3 \end{pmatrix} \in \mathbb{R}^4$ lin. indep?

$$\left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 2 & 0 & -4 & 0 \\ -3 & 4 & 14 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right) \sim \overset{G\text{-}J}{\dots} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \text{no free vars} \rightsquigarrow \text{YES!}$$

Ex Are $\begin{pmatrix}2\\1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\0\end{pmatrix}, \begin{pmatrix}5\\1\\4\\3\end{pmatrix} \in \mathbb{R}^4$ lin. indep ?

$$\left(\begin{array}{ccc|c} 2 & 1 & 5 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 4 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right) \sim \overset{G-J}{\cdots} \sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{matrix}x_3 \text{ free} \\ \text{var}\end{matrix} \rightsquigarrow \text{NO!}$$

$$\left[\text{In fact: } (-3)\begin{pmatrix}2\\1\\2\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\\2\\0\end{pmatrix} + 1\begin{pmatrix}5\\1\\4\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right]$$

Theorem Let $\underline{v}_1, \ldots, \underline{v}_n \in \mathbb{R}^n$, and let $A \in \mathbb{R}^{n \times n}$ be the matrix with columns $\underline{v}_1, \ldots, \underline{v}_n$. The vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly dependent iff A is singular (i.e., $\det(A) = 0$).

Pf The equation $\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}$ can be written as $A\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} = \underline{0}$.
Using the Invertible Matrix Theorem, this system has a non-trivial solution $\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} \neq \underline{0}$ iff A is singular. $\square$

Ex Vectors $\begin{pmatrix}-1\\3\\1\end{pmatrix}, \begin{pmatrix}5\\2\\5\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix} \in \mathbb{R}^3$ are lin. dependent

because $\begin{vmatrix} -1 & 5 & 4 \\ 3 & 2 & 5 \\ 1 & 5 & 6 \end{vmatrix} \underset{R_3+R_1}{\overset{R_2+3R_1}{=}} \begin{vmatrix} -1 & 5 & 4 \\ 0 & 17 & 17 \\ 0 & 10 & 10 \end{vmatrix} = 10 \cdot 17 \begin{vmatrix} -1 & 5 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \overset{R_2=R_3}{=} 0$

Basis and Dimension

Def A set $\{\underline{v}_1, \dots, \underline{v}_n\}$ of vectors is a basis for a v.sp. V if

(i) $\underline{v}_1, \dots, \underline{v}_n$ are lin. independent

(ii) $\mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n) = V$.

Ex Let $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is a basis for $\mathbb{R}^3$.

Indeed:

(i) $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are lin. indep, since

$$\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3 = \underline{0} \Rightarrow \begin{pmatrix} \alpha_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

(ii) $\underline{e}_1, \underline{e}_2, \underline{e}_3$ span $\mathbb{R}^3$, since an arbitrary vector $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in \mathbb{R}^3$ can be written as

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3 \in \mathrm{Span}(\underline{e}_1, \underline{e}_2, \underline{e}_3)$$

Ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$

lin. indep, since $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ ✓

spanning set: the spanning set test:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ REF} \rightsquigarrow \# \text{ leading vars} = 3, \text{ working in } \mathbb{R}^3$$

$\rightsquigarrow$ YES.

By defn. of spanning: an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ can be written:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b-c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

<u>Remark</u> A space can have many different bases.

<u>Ex</u> The matrices

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for $\mathbb{R}^{2\times 2}$.

(i) lin. indep:

If $a E_{11} + b E_{12} + c E_{21} + d E_{22} = O_{2\times 2}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = c = d = 0.$$

(ii) spanning set:

An arbitrary matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \alpha_{11} E_{11} + \alpha_{12} E_{12} + \alpha_{21} E_{21} + \alpha_{22} E_{22} \in \operatorname{Span}(E_{11}, E_{12}, E_{21}, E_{22})$$

<u>Ex</u> Standard bases for $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, P_n

- <u>the standard basis for $\mathbb{R}^n$</u> is $\{\underline{e}_1, \dots, \underline{e}_n\}$, where

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad \underline{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \text{ are the columns of } I_n.$$

- <u>the standard basis for $\mathbb{R}^{m\times n}$</u> is $\{E_{ij} \mid i = 1 \dots m \;\; j = 1 \dots n\}$

where E_{ij} is the matrix whose (i,j)-entry is 1 and all other entries are 0.

- <u>the standard basis for P_n</u> is $\{p_0, p_1, \dots, p_n\}$, where

$$p_i(t) = t^i, \quad i = 0, \dots, n.$$

<u>Theorem</u> (Steinitz)

If $\underline{u}_1, \dots, \underline{u}_m$ are linearly independent in $V = \mathrm{Span}(\underline{v}_1, \dots, \underline{v}_n)$ then $m \leq n$.

<u>Pf</u> See notes / optional pre-recorded lecture.

Corollary If a vector space V has a basis of n vectors then every basis of V has exactly n vectors.

[Any two bases of a vector space have the same cardinality]

Pf Supp. $\{\underline{v}_1, \dots, \underline{v}_n\}$ and $\{\underline{u}_1, \dots, \underline{u}_m\}$ are bases for V.

$$\left.\begin{array}{l} \text{Since } \underline{v}_1, \dots, \underline{v}_n \text{ are lin. indep, } \underline{u}_1, \dots, \underline{u}_m \text{ span } V \overset{Th}{\Longrightarrow} n \le m \\ \text{--- } \underline{u}_1, \dots, \underline{u}_m \text{ --- --, } \underline{v}_1, \dots, \underline{v}_n \text{ --- } \overset{Th}{\Longrightarrow} m \le n \end{array}\right\} \Rightarrow m = n.$$

$\square$

Def If a v.sp. V has a basis consisting of n vectors, we say that V has dimension n, and write

$$\dim V = n.$$

By convention, $\dim(\{\underline{0}\}) = 0$.

Ex $\dim \mathbb{R}^n = n$, $\dim \mathbb{R}^{m \times n} = mn$, $\dim P_n = n+1$

Ex Subspaces of $\mathbb{R}^3$:

- 0-dimensional: $\{\underline{0}\}$
- 1-dimensional: any subspace spanned by one non-zero vector, i.e., a line through the origin. $\mathrm{Span}(\underline{u})$, $\underline{u} \ne \underline{0}$
- 2-dimensional: any subspace spanned by two linearly independent vectors, i.e., a plane through the origin. $\mathrm{Span}(\underline{u}, \underline{v})$, $\underline{u}, \underline{v}$ lin. indep.
- 3-dimensional: $\mathbb{R}^3$.

Th Let V be a v.sp. with $\dim(V) = n$. Then:

(1) any set of n lin. indep. vectors spans V (so is a basis)

(2) any n vectors that span V are lin. indep. (so a basis)

Pf. (1) Let $\underline{v}_1, \ldots, \underline{v}_n$ be lin. indep.

Pick $\underline{v} \in V$. Since $\dim V = n$, the $n+1$ vectors

$\underline{v}, \underline{v}_1, \ldots, \underline{v}_n$ must be linearly dependent (by Steinitz).

so $\alpha_0 \underline{v} + \alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n = \underline{0}$, where $\alpha_0, \alpha_1, \ldots, \alpha_n$ are not all 0.

Note $\alpha_0 \neq 0$ (otherwise we would have $\underline{v}_1, \ldots, \underline{v}_n$ are lin. dep.)

$$\Rightarrow \quad \underline{v} = \left(-\frac{\alpha_1}{\alpha_0}\right) \underline{v}_1 + \cdots + \left(\frac{-\alpha_n}{\alpha_0}\right) \underline{v}_n \in \operatorname{Span}(\underline{v}_1, \ldots, \underline{v}_n).$$

But $\underline{v}$ was arbitrary, so $\operatorname{Span}(\underline{v}_1, \ldots, \underline{v}_n) = V$.

(2) Exercise. $\square$

Remark

A basis is
- a minimal spanning set
- a maximal linearly independent set.

Ex $\dim(\mathbb{R}^3) = 3$

(1) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$ cannot span $\mathbb{R}^3$, since it has $2 < \overset{\dim}{\overset{\shortparallel}{3}}$ vectors, and the size of any spanning set is $\geq \dim$.

(2) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$ because it contains 3 lin. indep. vectors

Check: $\begin{vmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 3 \cdot 6 \neq 0$ ✓

(3) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$ because the first three vectors already span $\mathbb{R}^3$ by (2), but it is not lin. indep, since it contains $4 > \underset{\dim}{\underset{=}{3}}$ vectors.