MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 2 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW2 due date.

Exercise 2. We are assuming that

$$\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}, \tag{1}$$

and we must show that $\mathbf{u}=\mathbf{v}$. By axiom (A4) there is an element $-\mathbf{w}$ with $\mathbf{w}+(-\mathbf{w})=\mathbf{0}$. Thus

$$\begin{aligned}\mathbf{u} &\overset{(A3)}{=} \mathbf{u}+\mathbf{0} \overset{(A4)}{=} \mathbf{u}+(\mathbf{w}+(-\mathbf{w})) \overset{(A2)}{=} (\mathbf{u}+\mathbf{w})+(-\mathbf{w}) \\ &\overset{(1)}{=} (\mathbf{v}+\mathbf{w})+(-\mathbf{w}) \overset{(A2)}{=} \mathbf{v}+(\mathbf{w}+(-\mathbf{w})) \overset{(A4)}{=} \mathbf{v}+\mathbf{0} \overset{(A3)}{=} \mathbf{v}.\end{aligned}$$

Exercise 3. (a) $H_1 = \left\{ (r,s,t)^T \,\middle|\, r,s,t\in\mathbb{R} \text{ and } 3r+s-2t=0 \right\}$ is a subspace of $\mathbb{R}^3$. To prove this, first note that $(0,0,0)^T\in H_1$, because $3\cdot 0+0-2\cdot 0=0$, so H_1 is non-empty. To prove closure under addition, we must take arbitrary vectors $\mathbf{x}_1=(r_1,s_1,t_1)^T\in H_1$ and $\mathbf{x}_2=(r_2,s_2,t_2)\in H_1$ and show that $\mathbf{x}_1+\mathbf{x}_2$ must be in H_1. Since $\mathbf{x}_1\in H_1$ and $\mathbf{x}_2\in H_1$, we have $3r_1+s_1-2t_1=0$ and $3r_2+s_2-2t_2=0$, so

$$\mathbf{x}_1+\mathbf{x}_2=\begin{pmatrix} r_1+r_2\\ s_1+s_2\\ t_1+t_2\end{pmatrix}$$

is also in H_1 because $3(r_1+r_2)+(s_1+s_2)-2(t_1+t_2)=(3r_1+s_1-2t_1)+(3r_2+s_2-2t_2)=0+0=0$. To prove closure under scalar multiplication, we must take an arbitrary vector $\mathbf{x}=(r,s,t)^T\in H_1$ and an arbitrary scalar α and prove that $\alpha\mathbf{x}$ is in H_1. Since $\mathbf{x}$ is in H_1, we have $3r+s-2t=0$, so

$$\alpha\mathbf{x}=\begin{pmatrix}\alpha r\\ \alpha s\\ \alpha t\end{pmatrix}$$

is in H_1 because $3\alpha r+\alpha s-2\alpha t=\alpha(3r+s-2t)=0$. We have shown that H_1 is non-empty and closed under addition and scalar multiplication, so H_1 is a subspace of $\mathbb{R}^3$.

(b) $H_2 = \left\{ (r+1,0,r)^T \,\middle|\, r\in\mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^3$ because it is not closed under scalar multiplication. For example, $\mathbf{x}=(1,0,0)^T$ is in H_2 (to see this take $r=0$), but $2\mathbf{x}=(2,0,0)^T$ is not in H_2 (because there is no r such that $(2,0,0)^T=(r+1,0,r)^T$).

(c) $H_3 = \left\{ (r,s,t)^T \,\middle|\, r,s,t\in\mathbb{R} \text{ and } r^2+s^2+t^2\le 1 \right\}$ is not a subspace of $\mathbb{R}^3$ because it is not closed under scalar multiplication. For example, $\mathbf{x}=(0,0,1)^T$ is in H_3 because $0^2+0^2+1^2\le 1$, but $2\mathbf{x}=(0,0,2)^T$ is not in H_3 because $0^2+0^2+2^2=4\not\le 1$.

Exercise 4. The zero function $\mathbf{0}$ is differentiable on $[a,b]$ and its derivative $\mathbf{0}'=\mathbf{0}$ is continuous on $[a,b]$. Thus $C^1[a,b]$ is not empty.

Let us now show that $C^1[a,b]$ is closed under addition. Take arbitrary $\mathbf{f}$ and $\mathbf{g}$ belonging to $C^1[a,b]$. Since $\mathbf{f}$ and $\mathbf{g}$ are differentiable on $[a,b]$, their sum $\mathbf{h}=\mathbf{f}+\mathbf{g}$ is also differentiable, because the sum of any two differentiable functions is differentiable. The derivative $\mathbf{h}'=(\mathbf{f}+\mathbf{g})'=\mathbf{f}'+\mathbf{g}'$ is continuous on $[a, b]$, because $\mathbf{f}'$ and $\mathbf{g}'$ are continuous and the sum of any two continuous functions is continuous. Thus $\mathbf{f} + \mathbf{g} = \mathbf{h} \in C^1[a, b]$.

It remains to show that $C^1[a, b]$ is closed under scalar multiplication. Take an arbitrary function $\mathbf{f} \in C^1[a, b]$ and an arbitrary scalar $\alpha \in \mathbb{R}$. Then $\mathbf{k} = \alpha\mathbf{f}$ is differentiable because a scalar multiple of any differentiable function is differentiable. Its derivative $\mathbf{k}' = (\alpha\mathbf{f})' = \alpha\mathbf{f}'$ is a scalar multiple of the continuous function $\mathbf{f}'$, so it is itself continuous. Thus $\alpha\mathbf{f} = \mathbf{k} \in C^1[a, b]$.

(Remark: Here I am assuming that you are familiar with the fact that sums and scalar multiples of continuous, respectively differentiable, functions, are themselves continuous, respectively differentiable. If you don't remember exactly *why* this is true, you may need to consult a proof in your calculus notes/textbook.)

Exercise 5. Let us first show that $U \cap V$ is a subspace of W. Since U and V are subspaces, they must contain $\mathbf{0}$. Therefore, $\mathbf{0}$ is in $U \cap V$, so $U \cap V$ is non-empty. To prove that $U \cap V$ is closed under addition, consider arbitrary vectors $\mathbf{w}_1, \mathbf{w}_2 \in U \cap V$. Since $\mathbf{w}_1, \mathbf{w}_2 \in U$ and U is closed under addition, we have $\mathbf{w}_1 + \mathbf{w}_2 \in U$. Similarly, $\mathbf{w}_1 + \mathbf{w}_2 \in V$ because V is closed under addition. Therefore $\mathbf{w}_1 + \mathbf{w}_2 \in U \cap V$. To prove that $U \cap V$ is closed under scalar multiplication, consider an arbitrary vector $v \in U \cap V$ and an arbitrary scalar α. Since both U and V are closed under scalar multiplication, we have $\alpha v \in U$ and $\alpha v \in V$. That is, $\alpha v \in U \cap V$ as required. Therefore, $U \cap V$ is indeed a subspace of W.

Now let us show that $U + V$ is a subspace of W. The zero vector can be written as $\mathbf{0} = \mathbf{0} + \mathbf{0}$, so it is an element of $U + V$ (because $\mathbf{0} \in V$ and $\mathbf{0} \in W$). Since U and V are both subspaces, they both contain $\mathbf{0}$. Since they are both closed under addition, they both contain $\mathbf{0} + \mathbf{0} = \mathbf{0}$. Now consider arbitrary vectors $\mathbf{w}_1, \mathbf{w}_2 \in U + V$ and an arbitrary scalar α. Then, by definition of $U + V$, there are vectors $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that

$$\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1 \quad \text{and} \quad \mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2.$$

Thus, by axioms (A1) and (A2), we have

$$\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2).$$

Since $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{v}_1 + \mathbf{v}_2 \in V$ (because U and V are closed under addition), it follows that $\mathbf{w}_1 + \mathbf{w}_2$ is an element of $U + V$. That is, $U + V$ is closed under addition. Moreover, by axiom (A5), we have

$$\alpha\mathbf{w}_1 = \alpha(\mathbf{u}_1 + \mathbf{v}_1) = \alpha\mathbf{u}_1 + \alpha\mathbf{v}_1.$$

Since U and V are closed under scalar multiplication, we have $\alpha\mathbf{u}_1 \in U$ and $\alpha\mathbf{v}_1 \in V$, and so it follows that $\alpha\mathbf{w}_1$ is an element of $U + V$. Therefore, $U + V$ is closed under scalar multiplication. This completes the proof that $U + V$ is a subspace of W.

Exercise 6. (a) This is true. To prove it, first note that $(0, 0, 0, 0)^T \in H_1$, so H_1 is not empty. Moreover, if $\mathbf{x}_1 = (r_1, s_1, t_1, u_1)^T \in H_1$ and $\mathbf{x}_2 = (r_2, s_2, t_2, u_2)^T \in H_1$ then $r_1 + s_1 - 3t_1 + 5u_1 = 0$ and $r_2 + s_2 - 3t_2 + 5u_2 = 0$ so

$$\mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} r_1 + r_2 \\ s_1 + s_2 \\ t_1 + t_2 \\ u_1 + u_2 \end{pmatrix}$$

is in H_1 because $(r_1 + r_2) + (s_1 + s_2) - 3(t_1 + t_2) + 5(u_1 + u_2) = (r_1 + s_1 - 3t_1 + 5u_1) + (r_2 + s_2 - 3t_2 + 5u_2) = 0 + 0 = 0$. Furthermore, if $\mathbf{x} = (r, s, t, u)^T \in H_1$ and α is a scalar then $r + s - 3t + 5u = 0$, so

$$\alpha \mathbf{x} = \begin{pmatrix} \alpha r \\ \alpha s \\ \alpha t \\ \alpha u \end{pmatrix}$$

in H_1 because $\alpha r + \alpha s - 3\alpha t + 5\alpha u = \alpha(r + s - 3t + 5u) = \alpha 0 = 0$. Thus H_1 is non-empty and closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^4$.

(b) This is also true. To prove it, first observe that H_2 is not empty (for example, the zero matrix is symmetric, so belongs to H_2). Suppose now that A and B belong to H_2. Then $A^T = A$ and $B^T = B$, and

$$(A + B)^T = A^T + B^T = A + B\,,$$

so $A + B$ is in H_2. Furthermore, if $A \in H_2$ and α is a scalar, then $A^T = A$ and

$$(\alpha A)^T = \alpha A^T = \alpha A\,,$$

so αA is in H_2. Thus H_2 is non-empty and closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^{n \times n}$.

(c) This is false. The set H_3 is not a subspace of $C[0, 1]$, because it does not contain the zero function.

Exercise 7. The definition of "subspace" guarantees that H is closed under addition and scalar multiplication, and since these operations are the same as those in V, we already know that they satisfy all of the vector space axioms (A_1), (A_2), (A_5), (A_6), (A_7) and (A_8) from lectures. We just need to check that (A_3) and (A_4) hold in H.

For (A_3), we must show that the zero vector of V is an element of H. Since H is non-empty, we can take any vector $\mathbf{v} \in H$ and multiply it by the scalar 0 to give $0\mathbf{v} = \mathbf{0}$ (this equality is from Lemma 4.2 in lectures, which you should have proved as a separate exercise). But H is closed under scalar multiplication, so $0\mathbf{v} = \mathbf{0}$ is indeed an element of H.

For (A_4), we must show that $-\mathbf{v}$ is in H whenever $\mathbf{v}$ is in H. However, we know from lectures (Lemma 4.2 again) that $-\mathbf{v} = (-1)\mathbf{v}$, so again we may conclude that $-\mathbf{v} \in H$ because H is closed under scalar multiplication.