MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 2

WebWork submission of **exercise marked** $(*)$ due:
11.59am on Wednesday 18 October 2023

You should also attempt all of the other exercises in order develop your mathematical reasoning and skill in constructing arguments and proofs; model solutions will be posted on QMPlus after the submission date.

Exercise $(*)$ 1. Solve WeBWork **Set 2** at:

`https://webwork.qmul.ac.uk/webwork2/MTH5112-2023/.`

Log in with your 'ah***' QMUL ID as username, and your student number as password, see Coursework 0 for further instructions.

Exercise 2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in a vector space V. Use the axioms of a vector space to prove the following 'cancellation law' for vector addition:

$$\text{if} \quad \mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w} \quad \text{then} \quad \mathbf{u}=\mathbf{v}.$$

Indicate which vector space axiom you are using at each step of your proof.

Exercise 3. Determine whether each of the following subsets of $\mathbb{R}^3$ is a subspace or not. Justify your answers carefully:

(a) $H_1 = \{(r,s,t)^T \mid r,s,t \in \mathbb{R} \text{ and } 3r+s-2t=0\}$.
(b) $H_2 = \{(r+1,0,r)^T \mid r \in \mathbb{R}\}$.
(c) $H_3 = \{(r,s,t)^T \mid r,s,t \in \mathbb{R} \text{ and } r^2+s^2+t^2 \leq 1\}$.

Remember: if you think that H_i is a subspace then you must prove that it is (non-empty and) closed under both addition and scalar multiplication; if you think that H_i is *not* a subspace then you must find a counterexample to one of the two closure axioms, i.e. either find $\mathbf{v},\mathbf{w} \in H_i$ such that $\mathbf{v}+\mathbf{w} \notin H_i$, or find $\mathbf{v} \in H_i$ and $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin H_i$.

Exercise 4. Consider the vector space $C[a,b]$ of continuous functions $f:[a,b] \to \mathbb{R}$ on a closed interval $[a,b]$, which we defined in lectures. Let $C^1[a,b]$ denote the subset of $C[a,b]$ consisting of all the functions in $C[a,b]$ that have a continuous derivative, i.e. in order for $f \in C[a,b]$ to be an element of $C^1[a,b]$, it must be differentiable on the interval $[a,b]$, and its derivative must be a continuous function. Explain why $C^1[a,b]$ is a subspace of $C[a,b]$.

Exercise 5. Let U and V be subspaces of a vector space W. Define the *intersection* of U and V by

$$U \cap V = \{\mathbf{w} \in W \mid \mathbf{w} \in U \text{ and } \mathbf{w} \in V\},$$

and the *sum* of U and V by

$$U+V = \{\mathbf{w} \in W \mid \mathbf{w} = \mathbf{u}+\mathbf{v} \text{ for some } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}.$$

Prove that $U \cap V$ and $U+V$ are both subspaces of W.

Exercise 6. Which of the following statements are true? Justify your answers carefully:

(a) $H_1 = \{(r,s,t,u)^T \mid r,s,t,u \in \mathbb{R} \text{ and } r+s-3t+5u=0\}$ is a subspace of $\mathbb{R}^4$.

(b) $H_2 = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is symmetric}\}$ is a subspace of $\mathbb{R}^{n\times n}$ (the vector space of all $n \times n$ matrices).

(c) $H_3 = \{\mathbf{f} \in C[0,1] \mid \mathbf{f}(1) = 1\}$ is a subspace of $C[0,1]$.

Exercise 7. Prove that if H is a subspace of a vector space V, then H itself a vector space, under the same "addition" and "scalar multiplication" as in V. (That is, prove Theorem 4.7 from lectures.)