Linear combinations and span

Def Let V be a v.sp. and $\underline{v}_1, \ldots, \underline{v}_n \in V$.
A linear combination of $\underline{v}_1, \ldots, \underline{v}_n$ is a vector of the form

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

Ex Linear combinations of $\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$ include

$$2\underline{v}_1 + 3\underline{v}_2 = 2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$(-5)\underline{v}_1 + \frac{1}{2}\underline{v}_2 = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix} = \begin{pmatrix} -9/2 \\ 1 \\ 13/2 \end{pmatrix}$$

$$\pi \underline{v}_1 + e \underline{v}_2$$

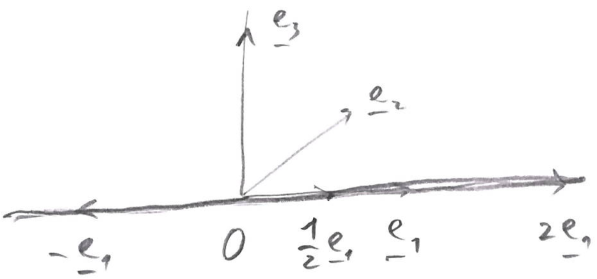
$$0 \cdot \underline{v}_1 + 0 \cdot \underline{v}_2 = \underline{0} \quad \longleftarrow \text{a trivial linear combination.}$$

Def Let V v.sp., $\underline{v}_1, \ldots, \underline{v}_n \in V$.
The span of $\underline{v}_1, \ldots, \underline{v}_n$ is the set of all linear combinations of $\underline{v}_1, \ldots, \underline{v}_n$, i.e.

$$\mathrm{Span}(\underline{v}_1, \ldots, \underline{v}_n) = \{ \alpha_1 \underline{v}_1 + \cdots + \alpha_n \underline{v}_n : \alpha_1, \ldots, \alpha_n \in \mathbb{R} \}.$$

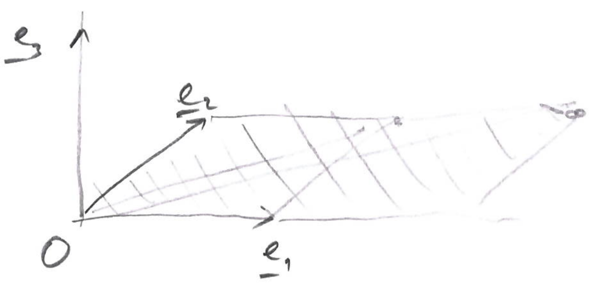
Ex Consider vectors $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

- $\mathrm{Span}(\underline{e}_1) = \{ \alpha_1 \underline{e}_1 : \alpha_1 \in \mathbb{R} \} = \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \alpha_1 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \alpha_1 \\ 0 \\ 0 \end{pmatrix} : \alpha_1 \in \mathbb{R} \right\}$



$\rightsquigarrow$ the x-axis.

- $\mathrm{Span}(\underline{e}_1, \underline{e}_2) = \{ \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 : \alpha_1, \alpha_2 \in \mathbb{R} \} = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R} \right\}$



$\rightsquigarrow$ the (x,y)-plane

- $\mathrm{Span}(\underline{e}_1, \underline{e}_2, \underline{e}_3) = \{ \alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \alpha_3 \underline{e}_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \}$

$$= \left\{ \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\} = \mathbb{R}^3 .$$

Def Let V v.sp., $S \subseteq V$ a subset.

Let Span(S) be the set of all linear combinations of vectors from S, i.e.,

$$\text{Span}(S) = \left\{ \sum_{i=1}^{n} \alpha_i \underline{v}_i : n \in \mathbb{N},\ \underline{v}_1, \dots, \underline{v}_n \in S,\ \alpha_1 \dots \alpha_n \in \mathbb{R} \right\}$$

Prop Let V v.sp. $S \subseteq V$

Then Span(S) is a subspace of V.

Pf Clearly $\underline{0} \in \text{Span}(S)$ for any S (the trivial lin. comb.).

So Span$(S) \neq \emptyset$.

(i) Span(S) is closed under addition.

Let $\underline{u}, \underline{v} \in \text{Span}(S)$; then

$\underline{u} = \alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n$ for some $\underline{u}_1, \dots, \underline{u}_n \in S$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

$\underline{v} = \beta_1 \underline{v}_1 + \dots + \beta_m \underline{v}_m$ — — — $\underline{v}_1, \dots, \underline{v}_m \in S$, $\beta_1 \dots \beta_m \in \mathbb{R}$.

$\Rightarrow$ $\underline{u} + \underline{v} = \alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n + \beta_1 \underline{v}_1 + \dots + \beta_m \underline{v}_m$ is again a linear combination of vectors from S, so $\underline{u} + \underline{v} \in \text{Span}(S)$.

(ii) Span(S) is closed under scalar mult.

Let $\lambda \in \mathbb{R}$, $\underline{u} \in \text{Span}(S)$; then

$\underline{u} = \alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n$, for some $\underline{u}_1, \dots, \underline{u}_n \in S$, $\alpha_1 \dots \alpha_n \in \mathbb{R}$

$\Rightarrow$ $\lambda \underline{u} = \lambda(\alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n) = (\lambda \alpha_1) \underline{u}_1 + \dots + (\lambda \alpha_n) \underline{u}_n$

is again a lin. comb. of vectors from S, so $\lambda \underline{u} \in \text{Span}(S)$.

$\square$

<u>Terminology</u> Let V be a v.sp., $\underline{v}_1, \ldots, \underline{v}_n \in V$ and let H be a subspace of V s.t.

$$H = \mathrm{Span}(\underline{v}_1, \ldots, \underline{v}_n)$$

[i.e., every vector in H can be written as a linear comb. of $\underline{v}_1, \ldots, \underline{v}_n$]

Then we can say either:

- $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a <u>spanning set</u> for H
- the set $\{\underline{v}_1, \ldots, \underline{v}_n\}$ <u>spans</u> H
- vectors $\underline{v}_1, \ldots, \underline{v}_n$ <u>span</u> H
- H is <u>spanned</u> by $\underline{v}_1, \ldots, \underline{v}_n$
- H is the <u>span</u> of $\underline{v}_1, \ldots, \underline{v}_n$.

Ex ① Is $S_1 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\4\end{pmatrix} \right\}$ a spanning set for $\mathbb{R}^3$?

Yes, because an arbitrary vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ can be written as

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} + 0\begin{pmatrix}1\\2\\4\end{pmatrix}$$

↳ this vector is redundant.

② Is $S_2 = \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ a spanning set for $\mathbb{R}^3$?

Q: Given an arbitrary $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$, can we find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

s.t. $$\begin{pmatrix}a\\b\\c\end{pmatrix} = \alpha_1\begin{pmatrix}1\\2\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\3\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\alpha_1 + \alpha_3\\ 2\alpha_1 + 3\alpha_2\\ \alpha_2 + \alpha_3\end{pmatrix}$$

Can we solve a linear system

$$\begin{aligned} \alpha_1 \quad\quad + \alpha_3 &= a \\ 2\alpha_1 + 3\alpha_2 \quad\quad &= b \\ \alpha_2 + \alpha_3 &= c \end{aligned}$$

in unknowns $\alpha_1, \alpha_2, \alpha_3$, FOR ALL $a, b, c \in \mathbb{R}$?

Coefficient matrix is

$$A = \begin{pmatrix}1 & 0 & 1\\ 2 & 3 & 0\\ 0 & 1 & 1\end{pmatrix}$$

Since $\det(A) \neq 0$ [check for CW], A is invertible

so the system $A\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ has a (unique) solution

$$\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = A^{-1}\begin{pmatrix}a\\b\\c\end{pmatrix} \implies \mathbb{R}^3 = \mathrm{Span}(S_2)$$

(3) Does $S_3 = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ span $\mathbb{R}^3$?

i.e., can every $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ be written as $\begin{pmatrix}a\\b\\c\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\1\\1\end{pmatrix}$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$?

The corresponding linear system is

$$\begin{aligned} \alpha_1 + \alpha_2 &= a \\ \alpha_2 &= b \\ \alpha_1 + \alpha_2 &= c \end{aligned} \quad \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{array}\right) \sim \cdots \overset{G-J}{\sim} \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & c-a \end{array}\right)$$

INCONSISTENT iff $c \neq a$.

so S_3 does NOT span $\mathbb{R}^3$.

We can only get $\begin{pmatrix}a\\b\\c\end{pmatrix}$ with $a = c$, so $\mathrm{Span}(S_3) \subsetneq \mathbb{R}^3$

$\left[\text{e.g. } \begin{pmatrix}1\\2\\3\end{pmatrix} \notin \mathrm{Span}(S_3) \right]$.

(4) Consider $p_1(t) = 2 + 3t + t^2$, $p_2(t) = 4 - t$, $p_3(t) = -1 \in P_2$

<u>Claim</u>: $P_2 = \mathrm{Span}(p_1, p_2, p_3)$.

We must show that an arbitrary $p \in P_2$, say

$$p(t) = a + bt + ct^2$$

can be written as

$$p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \quad \text{for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}.$$

$$a + bt + ct^2 = \alpha_1 (2 + 3t + t^2) + \alpha_2 (4 - t) + \alpha_3 (-1)$$

Compare the like powers of t:

$$\begin{aligned} a &= 2\alpha_1 + 4\alpha_2 - \alpha_3 \\ b &= 3\alpha_1 - \alpha_2 \\ c &= \alpha_1 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \alpha_1 &= c \\ \alpha_2 &= 3c - b \\ \alpha_3 &= 2c + 4(3c - b) - a \end{aligned}$$

Since we have a solution for any $a, b, c \in \mathbb{R}$,

we conclude $P_2 = \mathrm{Span}(p_1, p_2, p_3)$.

Theorem Let $A \in \mathbb{R}^{m \times n}$, and let $\underline{a}_1, \dots, \underline{a}_n \in \mathbb{R}^m$ be columns of A, and $\underline{b} \in \mathbb{R}^m$.

The system $A\underline{x} = \underline{b}$ is consistent iff $\underline{b} \in \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$.

Pf $\boxed{\Rightarrow}$ If the system is consistent, we have at least one solution

$$\underline{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n, \text{ i.e., } \underline{b} = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \underline{a}_1 + \dots + \alpha_n \underline{a}_n .$$

$\Rightarrow \underline{b} \in \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$.

$\boxed{\Leftarrow}$ Conversely, if $\underline{b} \in \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$, then $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t. $\underline{b} = \alpha_1 \underline{a}_1 + \dots + \alpha_n \underline{a}_n = A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, so $\underline{x} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is a solution $A\underline{x} = \underline{b}$. $\square$

Span Membership Test.

Given $\underline{a}_1, \dots, \underline{a}_n, \underline{b} \in \mathbb{R}^m$, decide whether

$$\underline{b} \in \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n).$$

- Let A be the $m \times n$ matrix with columns $\underline{a}_1, \dots, \underline{a}_n$.
- reduce the augmented matrix $(A \mid \underline{b})$ to REF.
- if consistent, $\underline{b} \in \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$.

  if inconsistent, $\underline{b} \notin \mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n)$.

Ex

(1) Is $\begin{pmatrix} 0 \\ 5 \\ 2 \\ 6 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right)$?

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 6 \end{array}\right) \overset{G-J}{\sim \ \cdots \ \sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\rightsquigarrow$ consistent $\rightsquigarrow$ YES!

(2) Is $\begin{pmatrix} 7 \\ 8 \\ 9 \\ 10 \end{pmatrix} \in \operatorname{Span}(\text{ — '' — })$?

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 8 \\ 1 & 1 & 3 & 9 \\ 1 & 2 & 4 & 10 \end{array}\right) \overset{G-J}{\sim \ \cdots \ \sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \vdots \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \vdots \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\rightsquigarrow$ inconsistent $\rightsquigarrow$ No!

(3) Determine $\operatorname{Span}(\text{ — '' — })$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y \\ 1 & 1 & 3 & z \\ 1 & 2 & 4 & t \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & 2 & z-x \\ 0 & 1 & 3 & t-x \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 2 & y \\ 0 & 0 & 1 & (z-x)/2 \\ 0 & 0 & 0 & t-x-y-\frac{(z-x)}{2} \end{array}\right)$$

Consistent $\iff$ $t - x - y - \frac{(z-x)}{2} = 0$, i.e.

$$\boxed{x + 2y + z - 2t = 0}$$

So $\operatorname{Span}(\text{— '' —}) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \ : \ x + 2y + z - 2t = 0 \right\}$

a proper subspace of $\mathbb{R}^4$.

Spanning Set Test Given $\underline{a}_1, \dots, \underline{a}_n \in \mathbb{R}^m$, decide whether

$$\mathrm{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^m .$$

- form the $m \times n$ matrix A with columns $\underline{a}_1, \dots, \underline{a}_n$.
- reduce A to REF
- if the number of leading 1s $= m \rightsquigarrow$ YES

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad < m \rightsquigarrow$ NO.

Ex ① $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$ is not a spanning set for $\mathbb{R}^4$

because the REF (above) has 3 leading 1s $(3<4)$

② $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix} \right\}$ IS a spanning set for $\mathbb{R}^4$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 5 \\ 0 & 1 & 2 & 0 & 6 \\ 1 & 1 & 3 & 1 & 5 \\ 1 & 2 & 4 & 3 & 6 \end{pmatrix} \sim \overset{\text{G-J}}{\cdots} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$ has 4 leading 1s

Properties of Span

Prop Let V be a v.sp.

(i) For $S \subseteq V$, $S \subseteq \mathrm{Span}(S)$

(ii) If $S \subseteq T \subseteq V$, then $\mathrm{Span}(S) \subseteq \mathrm{Span}(T)$.

(iii) If $S \subseteq V$, then $\mathrm{Span}(\mathrm{Span}(S)) = \mathrm{Span}(S)$.

Pf (i) For any $\underline{v} \in S$, $\underline{v} = 1 \cdot \underline{v}$ is a lin. comb. of vectors from S $\Rightarrow$ $\underline{v} \in \mathrm{Span}(S)$. Hence $S \subseteq \mathrm{Span}(S)$.

(ii) Assume $S \subseteq T$.

If $\underline{v} \in \mathrm{Span}(S) \Rightarrow \underline{v} = \sum_{i=1}^{n} \alpha_i \underline{v_i}$, for some $\underline{v_1}, \dots, \underline{v_n} \in S$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$

Since $S \subseteq T$, we have that also $\underline{v_1}, \dots, \underline{v_n} \in T$

so $\underline{v}$ is a linear comb. of elements of T $\Rightarrow$ $\underline{v} \in \mathrm{Span}(T)$.

Hence $\mathrm{Span}(S) \subseteq \mathrm{Span}(T)$.

(iii) By (i), $S \subseteq \mathrm{Span}(S)$

by (ii), $\mathrm{Span}(S) \subseteq \mathrm{Span}(\mathrm{Span}(S))$.

Hence, it suffices to show $\mathrm{Span}(\mathrm{Span}(S)) \subseteq \mathrm{Span}(S)$.

Let $\underline{v} \in \mathrm{Span}(\mathrm{Span}(S))$; then $\underline{v} = \sum_{i=1}^{m} \lambda_i \underline{v_i}$, for some $m \in \mathbb{N}$, $\underline{v_1}, \dots, \underline{v_m} \in \mathrm{Span}(S)$, $\lambda_1, \dots, \lambda_m \in \mathbb{R}$

For each i, $\underline{v_i} = \sum_{j=1}^{n_i} \alpha_{ij} \underline{v_{ij}}$ for some $n_i \in \mathbb{N}$, $v_{ij} \in S$, $\alpha_{ij} \in \mathbb{R}$.

Thus, $\underline{v} = \sum_{i=1}^{m} \lambda_i \underline{v_i} = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{n_i} \alpha_{ij} \underline{v_{ij}} = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (\lambda_i \alpha_{ij}) \underline{v_{ij}}$

is a linear comb. of vectors from S $\Rightarrow$ $\underline{v} \in \mathrm{Span}(S)$. $\square$

<u>Corollary</u> If S is a spanning set for a v.sp. V and $T \supseteq S$, then T ______________ .

<u>Pf</u> Assume $S \subseteq T$; by Prop and assumption that S is spanning

$$V = \mathrm{Span}(S) \subseteq \mathrm{Span}(T) \subseteq V \implies \mathrm{Span}(T) = V \quad \square$$

<u>Prop</u> (Exchange Lemma) Let V v.sp., $S \subseteq V$, $\underline{a}, \underline{b} \in V$

If $\underline{a} \in \mathrm{Span}(S \cup \{\underline{b}\}) \setminus \mathrm{Span}(S)$, then $\underline{b} \in \mathrm{Span}(S \cup \{\underline{a}\})$.

<u>Pf</u> Supp. $\underline{a} \in \mathrm{Span}(S \cup \{\underline{b}\}) \setminus \mathrm{Span}(S)$.

Since $\underline{a} \in \mathrm{Span}(S \cup \{\underline{b}\})$, $\exists\ \underline{v}_1, \dots, \underline{v}_n \in S$, $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{R}$

s.t. $\underline{a} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n + \beta \underline{b}$.

Note that $\beta \neq 0$; otherwise $\underline{a} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n \in \mathrm{Span}(S)$.

Hence $\underline{b} = \frac{1}{\beta}\underline{a} + \left(\frac{-\alpha_1}{\beta}\right)\underline{v}_1 + \dots + \left(\frac{-\alpha_n}{\beta}\right)\underline{v}_n \in \mathrm{Span}(S \cup \{\underline{a}\})$.

$\square$

Ex Let $S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \right\}$, $S_2 = \left\{ \begin{pmatrix} -2 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$.

Prove: $\mathrm{Span}(S_1) = \mathrm{Span}(S_2)$.

Need to check: (i) $\mathrm{Span}(S_2) \subseteq \mathrm{Span}(S_1)$

& (ii) $\mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$.

(i) By Prop., enough to verify $S_2 \subseteq \mathrm{Span}(S_1)$.

[because then: $S_2 \subseteq \mathrm{Span}(S_1)$ / Span

$\Rightarrow \mathrm{Span}(S_2) \subseteq \mathrm{Span}(\mathrm{Span}(S_1)) = \mathrm{Span}(S_1)$.]

In other words, we need to show $\begin{pmatrix} -2 \\ -6 \\ 0 \end{pmatrix} \in \mathrm{Span}(S_1)$ & $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \in \mathrm{Span}(S_1)$.

Apply span membership test to both at once

$$\left(\begin{array}{ccc|c|c} 1 & 0 & 2 & -2 & 1 \\ 2 & 1 & 5 & -6 & 1 \\ -1 & 1 & -1 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|c|c} 1 & 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow$$ both systems are consistent $\rightsquigarrow$ YES!

(ii) Similarly, verify that $S_1 \subseteq \mathrm{Span}(S_2) \rightsquigarrow$ CW.

Intersection and sum of subspaces

Prop Let L and M be subspaces of a v.sp. V. Then $L \cap M$ is also a subspace of V.

Pf Since L, M are subspaces, have $\underline{0} \in L$ & $\underline{0} \in M$

$\Rightarrow \underline{0} \in L \cap M$, so $L \cap M \neq \phi$.

(i) $L \cap M$ is closed under addition

Let $\underline{u}, \underline{v} \in L \cap M$, i.e., $\underline{u}, \underline{v} \in L$ & $\underline{u}, \underline{v} \in M$.

$$\left.\begin{array}{l} L \text{ subspace} \Rightarrow \underline{u} + \underline{v} \in L \\ M \text{ subspace} \Rightarrow \underline{u} + \underline{v} \in M \end{array}\right) \Rightarrow \underline{u} + \underline{v} \in L \cap M. \quad ✓$$

(ii) $L \cap M$ is closed under scalar mult.

Let $\alpha \in \mathbb{R}$, $\underline{u} \in L \cap M$. Hence $\underline{u} \in L$ & $\underline{u} \in M$.

$$\left.\begin{array}{l} L \text{ subspace} \Rightarrow \alpha \underline{u} \in L \\ M \text{ --- } \Rightarrow \alpha \underline{u} \in M \end{array}\right) \Rightarrow \alpha \underline{u} \in L \cap M. \quad ✓$$

$\square$

Def Let L, M be subspaces of a v.sp. V. Their <u>sum</u> is the subspace

$$L + M = \text{Span}(L \cup M)$$

Prop With above notation

$$L + M = \{ \underline{u} + \underline{v} : \underline{u} \in L, \underline{v} \in M \}.$$

Pf See CW.