MTH5112 Linear Algebra I
MTH5212 Applied Linear Algebra

COURSEWORK 1 — SOLUTIONS

Exercise (∗) 1. The solutions will appear on WeBWork after CW1 due date.

Exercise 2. Write $A = (a_{ij})$ and $B = (b_{ij})$, meaning (as in lectures) that a_{ij} and b_{ij} denote the (i, j)-entries of A and B, respectively.

(a) The matrices $(\alpha A)^T$ and $\alpha(A^T)$ have the same size because A is a square matrix. We need to check that they have the same entries. The (i, j)-entry of αA is αa_{ij}, so the (i, j)-entry of $(\alpha A)^T$ is αa_{ji}, which in turn is equal to the (i, j)-entry of $\alpha(A^T)$. Thus $(\alpha A)^T$ and $\alpha(A^T)$ have the same entries, so it follows that $(\alpha A)^T = \alpha(A^T)$.

(b) Again, since $(A + B)^T$ and $A^T + B^T$ have the same size we only need to check that they have the same entries. To see this, note that the (i, j)-entry of $A + B$ equals $a_{ij} + b_{ij}$, so the (i, j)-entry of $(A + B)^T$ equals $a_{ji} + b_{ji}$, which in turn is equal to the (i, j)-entry of $A^T + B^T$. Thus $(A + B)^T$ and $A^T + B^T$ have the same entries as claimed, and it follows that $(A + B)^T = A^T + B^T$.

Exercise 3.

(a) Using properties of matrix addition and multiplication, we find that

$$(A + B)^2 = (A + B)(A + B) = A^2 + BA + AB + B^2 .$$

Thus $(A + B)^2 = A^2 + 2AB + B^2$ if and only if $BA = AB$, i.e. if and only if A and B commute.

(b) We need to show that $I - A + A^2$ is the inverse of $I + A$ (under the assumption $A^3 = 0$). That is, we must show that the matrices $(I + A)(I - A + A^2)$ and $(I - A + A^2)(I + A)$ are both equal to the identity matrix I. Since $A^3 = 0$, we have

$$(I + A)(I - A + A^2) = I - A + A^2 + A - A^2 + A^3 = I + A^3 = I + 0 = I$$

and

$$(I - A + A^2)(I + A) = I - A + A^2 + A - A^2 + A^3 = I + A^3 = I + 0 = I ,$$

which is what we wanted. Therefore, $(I + A)^{-1} = I - A + A^2$. In particular, we see that $I + A$ is invertible because we have just written down its inverse!

(c) Since we are assuming that A is invertible, i.e. that A^{-1} exists, it at least makes sense to consider the possibility that the inverse of A^T is equal to $(A^{-1})^T$. To show that this is actually *true*, we must show that $A^T(A^{-1})^T$ and $(A^{-1})^T A^T$ are both equal to the identity matrix. Using the property in the hint, we find that

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I .$$

Thus A^T is invertible with inverse $(A^{-1})^T$, that is

$$(A^T)^{-1} = (A^{-1})^T .$$

Exercise 4. (a) Write $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. Since A and B are diagonal, we have $a_{ij} = 0$ and $b_{ij} = 0$ whenever $i \neq j$. Now write $AB = (c_{ij})_{n\times n}$, i.e. let c_{ij} denote the (i, j)-entry of AB. We are trying to show that AB is diagonal, so we must show that $c_{ij} = 0$ whenever $i \neq j$. The definition of matrix multiplication says that

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Now, in this notation we have $a_{ik} = 0$ whenever $i \neq k$, and $b_{kj} = 0$ whenever $j \neq k$; so the only way that one of the terms $a_{ik}b_{kj}$ in above sum can be *non-zero* is if both i and j are equal to k (because both a_{ik} and b_{kj} would have to be non-zero). In particular, i and j have to be equal to each other! If they are not, then $c_{ij} = 0$, which is what we were trying to show.

We also need to show that A and B commute. Write $BA = (d_{ij})_{n\times n}$. By the above proof, we know that BA is also diagonal (i.e. we could have just interchanged the roles of A and B in the above proof). Since both AB and BA are diagonal, we just need to show that their diagonal entries, i.e. those with $i = j$, are equal. That is, we must show that $c_{ii} = d_{ii}$ for all $i \in \{1, \dots, n\}$. We have

$$c_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki},$$

and in order for a term $a_{ik}b_{ki}$ in this sum to be non-zero, we need both a_{ik} and b_{ki} to be non-zero, so we need $k = i$. Therefore,

$$c_{ii} = a_{ii}b_{ii}.$$

But if we swap the roles of A and B in this calculation, we find that

$$d_{ii} = b_{ii}a_{ii} = a_{ii}b_{ii}.$$

Since this argument did not depend on the value of i, we have shown that $c_{ii} = d_{ii}$ for all i, which is what we wanted.

(b) Let's use the same notation $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $AB = (c_{ij})_{n\times n}$ as in part (a). We are assuming that A and B are upper triangular, i.e. that $a_{ij} = 0$ and $b_{ij} = 0$ whenever $i > j$. We must show that AB is upper triangular, i.e. that $c_{ij} = 0$ whenever $i > j$. From the definition of matrix multiplication, we can write

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj}.$$

Since A is upper triangular, we have $a_{ik} = 0$ whenever $i > k$; but we are also assuming that $i > j$ (because we are trying to show that $c_{ij} = 0$ in this case), so in the sum $\sum_{k=1}^{j} a_{ik}b_{kj}$ above we have $k \leq j < i$ and hence all of the a_{ik} in this sum are 0. Similarly, in the second sum $\sum_{k=j+1}^{n} a_{ik}b_{kj}$ we have $k < j$ (because k starts from $j + 1$ in this sum) and hence $b_{kj} = 0$ because B is upper triangular. Combining these last two observations, we conclude that when $i > j$ we have

$$c_{ij} = \sum_{k=1}^{j} \underbrace{a_{ik}}_{=0} b_{kj} + \sum_{k=j+1}^{n} a_{ik} \underbrace{b_{kj}}_{=0} = 0\,,$$

which means that $AB = (c_{ij})$ is indeed upper triangular.

(c) Two upper triangular matrices will not necessarily commute. Here is a counterexample. If

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{but} \quad BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 5. (a) We are assuming that A is symmetric, i.e. that $A^T = A$, and we must prove that BAB^T is symmetric, i.e. that $(BAB^T)^T = BAB^T$. By Proposition 2.12(4) in the lecture notes (which says that $(CD)^T = D^TC^T$ for matrices C and D), we have $(BAB^T)^T = (B^T)^T A^T B^T = BA^TB^T$. Since A is symmetric, this equals BAB^T, which is what we wanted.

(b) In general, we have $(AB)^T = B^TA^T$. If A and B are symmetric, it follows that $(AB)^T = BA$. This equals AB if and only if A and B commute (by definition of "commute").

(c) We are assuming that $AB = I$, and we are trying to prove that also $BA = I$. This means that B is invertible (with inverse A). By the Invertible Matrix Theorem, to prove that B is invertible, we can instead prove that that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. But if $B\mathbf{x} = \mathbf{0}$, then

$$\mathbf{x} = I\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0},$$

so indeed the only solution of $B\mathbf{x} = \mathbf{0}$ is the trivial solution. Hence, B is invertible, but we still need to show that A is the inverse of B, i.e. that $BA = I$ (we already know that $AB = I$, by assumption). Let C denote the inverse of B, so that $BC = I = CB$. Then, in particular, $CB = AB$ (because both are equal to I) and so part (d) gives $A = C$, i.e. A is the inverse of B. (Alternatively, observe that $BA = BAI = BA(BC) = B(AB)C = BIC = BC = I$.)

(d) Multiplying both sides of the equation $AB = AC$ on the left by A^{-1} (which we are assuming exists) gives $A^{-1}AB = A^{-1}AC$, i.e. $IB = IC$, i.e. $B = C$ as required.

Exercise 6. Matlab code:

```
A=[2 4; -6 0]

A =

     2     4
    -6     0

B=[1 -5; -3 2]

B =

     1    -5
    -3     2

A-B

ans =
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     1     9
    -3    -2

1/2*A-3*B

ans =

    -2    17
     6    -6

M=[1 1 1 -3 -2; 2 3 0 -4 1; -3 -4 -1 6 -1]

M =

     1     1     1    -3    -2
     2     3     0    -4     1
    -3    -4    -1     6    -1

rref(M)

ans =

     1     0     3     0     3
     0     1    -2     0     1
     0     0     0     1     2

C=inv(A)

C =

         0   -0.1667
    0.2500    0.0833

C*A

ans =

     1     0
     0     1

A*C

ans =

     1     0
     0     1
```