VECTOR SPACES

<u>Def</u> A <u>vector space</u> is a non-empty set V with two operations

(1) <u>addition</u> $\quad V \times V \to V$

$$(\underline{u}, \underline{v}) \longmapsto \underline{u} + \underline{v}$$

(2) <u>scalar multiplication</u> $\quad \mathbb{R} \times V \to V$

$$(\alpha, \underline{u}) \longmapsto \alpha \underline{u}$$

such that the following axioms hold <u>for all</u> $\underline{u}, \underline{v}, \underline{w} \in V$, $\alpha, \beta \in \mathbb{R}$

(C1) $\underline{u} + \underline{v} \in V$ [V is closed under addition]

(C2) $\alpha \underline{u} \in V$ [V is closed under scalar multiplication].

(A1) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ [addition is commutative]

(A2) $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ [addition is associative]

(A3) there is an element $\underline{0}$ ("zero") in V s.t.
for any $\underline{u} \in V$, $\underline{0} + \underline{u} = \underline{u}$.

(A4) for each $\underline{u} \in V$, there is an element "$-\underline{u}$" in V s.t.
$\underline{u} + (-\underline{u}) = \underline{0}$ [additive inverse].

(A5) $\alpha(\underline{u} + \underline{v}) = \alpha \underline{u} + \alpha \underline{v}$

(A6) $(\alpha + \beta)\underline{u} = \alpha \underline{u} + \beta \underline{u}$

(A7) $(\alpha \beta)\underline{u} = \alpha(\beta \underline{u})$

(A8) $1\,\underline{u} = \underline{u}$

The elements of V are called <u>vectors</u>.

Examples of vector spaces

Notation: we write

$$\mathbb{R}^{m\times n}$$

for the set of $m\times n$ real matrices, and

$$\mathbb{R}^n = \mathbb{R}^{n\times 1}$$

for the set of column vectors of length n

1 Let $V = \mathbb{R}^{m\times n}$, the set of $m\times n$ real matrices.

Last week, we discussed:

- addition of matrices:

  for $A=(\alpha_{ij})$, $B=(\beta_{ij}) \in \mathbb{R}^{m\times n}$, $A+B = (\alpha_{ij}+\beta_{ij}) \in \mathbb{R}^{m\times n}$

- scalar multiplication:

  for $\alpha \in \mathbb{R}$, $A=(\alpha_{ij}) \in \mathbb{R}^{m\times n}$, $\alpha A = (\alpha\, \alpha_{ij}) \in \mathbb{R}^{m\times n}$

We already verified that $\mathbb{R}^{m\times n}$ with those operations satisfies all axioms (A1) – (A8), so

$$\mathbb{R}^{m\times n} \text{ is a vector space,}$$

we call it the space of $m\times n$ matrices.

(2) Let $V = \mathbb{R}^n$ (column vectors of length n).

addition: for $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\underline{u} + \underline{v} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \in \mathbb{R}^n$

scalar mult: for $\alpha \in \mathbb{R}$, $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\alpha \underline{u} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix} \in \mathbb{R}^n$

This is a special case of the previous example, so it satisfies axioms (A1) - (A8), hence

$\mathbb{R}^n$ is a vector space,

the space of column vectors of length n.

(3) Let P_n be the set of polynomials (in variable t) with real coefficients of degree at most n.

An element $p \in P_n$ is an expression of the form

$$p(t) = a_0 + a_1 t + \dots + a_n t^n,$$

for some $a_0, a_1, \dots a_n \in \mathbb{R}$.

Addition of polynomials: if $q(t) = b_0 + b_1 t + \dots + b_n t^n \in P_n$

$p + q$

is the polynomial

$$(p+q)(t) = (a_0 + b_0) + (a_1 + b_1) t + \dots + (a_n + b_n) t^n \in P_n$$

addition in P_n addition in $\mathbb{R}$.

Scalar multiplication: For $\alpha \in \mathbb{R}$, $p \in P_n$ as above,

$$\alpha p$$

is the polynomial

$$(\alpha p)(t) = (\alpha a_0) + (\alpha a_1)t + \dots + (\alpha a_n)t^n \in P_n$$

Let us verify that P_n is a vector space

(C1), (C2) hold by definition of addition and scalar mult.

(A1): $(p+q)(t) \overset{\text{def.}}{=} (a_0+b_0) + (a_1+b_1)t + \dots + (a_n+b_n)t^n$

$\left(\begin{array}{l}\text{addition in } \mathbb{R} \\ \text{is commutative}\end{array}\right) \quad = (b_0+a_0) + (b_1+a_1)t + \dots + (b_n+a_n)t^n$

$$\overset{\text{def}}{=} (q+p)(t).$$

$\Rightarrow \quad p+q = q+p \quad$ ✓

(A2) similar to (A1) [using associativity of + in $\mathbb{R}$].

(A3) the "zero vector" in P_n is the zero polynomial $\underline{0} \in P_n$

defined by $\underline{0}(t) = 0 + 0 \cdot t + \dots + 0 \cdot t^n$

We have

$$(\underline{0}+p)(t) = (0+a_0) + (0+a_1)t + \dots + (0+a_n)t^n \overset{0 \text{ in } \mathbb{R}}{=}$$

$$= a_0 + a_1 t + \dots + a_n t^n = p(t).$$

$\Rightarrow \quad \underline{0} + p = p \quad$ ✓

(A4) Define $(-p) = (-1)p$, i.e.,

$$(-p)(t) = (-a_0) + (-a_1)t + \dots + (-a_n)t^n$$

Then for any $p \in P_n$,

$$(p + (-p))(t) = (a_0 + (-a_0)) + (a_1 + (-a_1))t + \dots + (a_n + (-a_n))t^n$$

$$= 0 + 0 \cdot t + \dots + 0 \cdot t^n = \underline{0}$$

$\Rightarrow p + (-p) = \underline{0}$ ✓

(A5) - (A8) Exercise.

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(4) Let $C[a,b]$ denote the set of continuous real-valued functions defined on the closed interval $[a,b] \subseteq \mathbb{R}$, i.e.

$$C[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b] \}$$

Addition and scalar multiplication on $C[a,b]$ are defined "pointwise"

For $f, g \in C[a,b]$

$$(f+g)(x) = f(x) + g(x), \quad \text{for all } x \in [a,b]$$

addition in $C[a,b]$ addition in $\mathbb{R}$.

Given $\alpha \in \mathbb{R}$, $f \in C[a,b]$

$$(\alpha f)(x) = \alpha f(x), \quad \text{for all } x \in [a,b].$$

scalar mult in $C[a,b]$ mult. in $\mathbb{R}$.

Let us verify that $C[a,b]$ is a vector space.

(C1) calculus: the sum of two continuous functions on $[a,b]$ is again a — — — — — .

(C2) calculus: the product of continuous functions on $[a,b]$ is again — — — — —

$\rightsquigarrow$ a constant times a continuous function on $[a,b]$ is — — — — — — .

(A1) $$(f+g)(x) \overset{\text{def}}{=} f(x)+g(x) \overset{+ \text{ in } \mathbb{R} \text{ is commutative}}{=} g(x)+f(x) \overset{\text{def}}{=} (g+f)(x)$$

for all $x \in [a,b]$, so $f+g = g+f$

(A2) similar

(A3) the "zero vector" is the constant function

$$\underline{0} : [a,b] \to \mathbb{R}$$
$$\underline{0}(x) = 0 \quad \text{for all } x \in [a,b].$$

We have

$$(\underline{0}+f)(x) \overset{\text{def}}{=} \underline{0}(x)+f(x) = 0+f(x) = f(x), \text{ for all } x \in [a,b]$$

$\Rightarrow \underline{0}+f = f$. ✓

(A4) For $f \in C[a,b]$, let $(-f) : [a,b] \to \mathbb{R}$ be

$$(-f)(x) = -(f(x)) \quad [\text{it is in } C[a,b]]$$

Then

$$(f+(-f))(x) = f(x)+(-f)(x) = f(x)+(-f(x)) = 0 = \underline{0}(x),$$
for all $x \in [a,b]$

$\Rightarrow f+(-f) = \underline{0}$ ✓

(A5) - (A8) Exercise.

Theorem Let V be a vector space, and $\underline{u}, \underline{v} \in V$

Then:
(a) $0\,\underline{u} = \underline{0}$

(b) $\alpha\,\underline{0} = \underline{0}$ for $\alpha \in \mathbb{R}$

(c) if $\underline{u} + \underline{v} = \underline{0}$, then $\underline{v} = (-\underline{u})$
[the additive inverse is unique]

(d) $-\underline{u} = (-1)\,\underline{u}$.

Pf (a) $0\,\underline{u} = (0+0)\,\underline{u} \overset{(A6)}{=} 0\,\underline{u} + 0\,\underline{u}$ / using (A4) add $-(0\,\underline{u})$ to both sides

$$\Rightarrow \underbrace{0\,\underline{u} + (-(0\,\underline{u}))}_{\overset{(A4)}{=}\ \underline{0}} = (0\,\underline{u} + 0\,\underline{u}) + (-(0\,\underline{u})) \overset{(A2)}{=} 0\,\underline{u} + (0\,\underline{u} + -(0\,\underline{u}))$$

$$\overset{(A4)}{=} 0\,\underline{u} + \underline{0} \overset{(A3)}{=} 0\,\underline{u}$$

$\Rightarrow 0\,\underline{u} = \underline{0}$ ✓

(b), (c), (d) exercise. □

Remark We defined vector spaces over $\mathbb{R}$, i.e. the scalars are real numbers.
The theory works well over an arbitrary field of scalars.

e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$

$\hookrightarrow$ Coding Theory.

Subspaces

<u>Def</u> Let V be a vector space

Let H be a non-empty subset of V $\quad [\phi \subsetneq H \subseteq V]$

We say that <u>H is a subspace of V</u> if:

(i) for all $\underline{u}, \underline{v} \in H$, we have $\underline{u} + \underline{v} \in H$

(ii) for all $\alpha \in \mathbb{R}$, $\underline{u} \in H$, we have $\alpha \underline{u} \in H$.

[A subspace of a vector space is a non-empty subset closed under addition and scalar multiplication]

<u>Theorem</u> Let H be a subspace of a v. sp. V
Then, with addition and scalar multiplication inherited from V, H is a vector space itself.

<u>Pf</u> CW.

<u>Remark</u> It is much easier to show that some structure H is a vector space by checking that it is a subspace of some other vector space V:
we only check two closure properties as opposed to 10 axioms.

<u>Example</u> Let V be a v. sp.
Then $\{\underline{0}\}$ and V are subspaces of V

$\hookrightarrow$ <u>the zero subspace</u>.

These are the <u>trivial subspaces</u>, all other subspaces are called <u>proper subspaces</u>.

Pf Let $H = \{\underline{0}\}$. We need to verify (i), (ii) from Defn.

(i) if $\underline{u}, \underline{v} \in H$, then $\underline{u} = \underline{v} = \underline{0}$, so $\underline{u} + \underline{v} = \underline{0} + \underline{0} = \underline{0} \in H$ ✓

(ii) if $\alpha \in \mathbb{R}$, $\underline{u} \in H$, then $\underline{u} = \underline{0}$, so $\alpha \underline{u} = \alpha \underline{0} \underset{\text{Thm.}}{=} \underline{0} \in H$ ✓

So H is a subspace of V.

Similarly, V is a subspace of itself since it satisfies (C1), (C2).

Example Let $L = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R},\ x = y = z \right\} \subseteq \mathbb{R}^3$.

Then L is a subspace of $\mathbb{R}^3$.

Pf Note $L \neq \emptyset$, because $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in L$.

(i) Supp. $\underline{u}, \underline{v} \in L$. Then $\underline{u} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$, $\underline{v} = \begin{pmatrix} b \\ b \\ b \end{pmatrix}$ for some $a, b \in \mathbb{R}$.

So $\underline{u} + \underline{v} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} + \begin{pmatrix} b \\ b \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a+b \\ a+b \end{pmatrix} \in L$, so L is closed under addition

(ii) Supp. $\alpha \in \mathbb{R}$, $\underline{u} \in L$. Then $\underline{u} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}$ for some $a \in \mathbb{R}$,

so $\alpha \underline{u} = \alpha \begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha a \\ \alpha a \end{pmatrix} \in L$, so L is closed under scalar mult.

(i) & (ii) $\Rightarrow$ L is a subspace of $\mathbb{R}^3$.

Example Let $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R},\ x - y + 3z = 0 \right\} \subseteq \mathbb{R}^3$.

Then P is a subspace of $\mathbb{R}^3$.

Pf Note that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in P$ so $P \neq \emptyset$.

(i) Let $\underline{u}, \underline{v} \in P$

Then $\underline{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ with $x_1 - y_1 + 3z_1 = 0$ for some $x_1, y_1, z_1 \in \mathbb{R}$

$\underline{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ with $x_2 - y_2 + 3z_2 = 0$ - - - $x_2, y_2, z_2 \in \mathbb{R}$

then $\underline{u} + \underline{v} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$, but is this vector in P?

We need to verify that $(x_1 + x_2) - (y_1 + y_2) + 3(z_1 + z_2) = 0$

Indeed, $(x_1 + x_2) - (y_1 + y_2) + 3(z_1 + z_2) = (x_1 - y_1 + 3z_1) + (x_2 - y_2 + 3z_2)$

$= 0 + 0 = 0$ ✓

So $\underline{u} + \underline{v} \in P$ hence P is closed under addition.

(ii) Let $\alpha \in \mathbb{R}$, $\underline{u} \in P$. Then $\underline{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $x - y + 3z = 0$,

and $\alpha \underline{u} = \alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \rightsquigarrow$ is this vector in P?

We need to verify that $(\alpha x) - (\alpha y) + 3(\alpha z) = 0$.

Indeed, $\alpha x - \alpha y + 3\alpha z = \alpha(x - y + 3z) = \alpha \cdot 0 = 0$ ✓

So $\alpha \underline{u} \in P$ and P is closed under scalar mult.

(i) & (ii) $\rightsquigarrow$ P is a subspace of $\mathbb{R}^3$.

<u>Non-Example</u> $H = \left\{ \begin{pmatrix} x^2 \\ y \\ x \end{pmatrix} : x, y \in \mathbb{R} \right\}$ is NOT a subspace of $\mathbb{R}^3$

To show that H is not a subspace, we must show that either (i) fails or (ii) fails, i.e., that either H is not closed under addition, or H — — — — — scalar mult.

We find a <u>counterexample</u> to (ii)

Consider $\underline{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in H$ (since $1^2 = 1$)

However, $2\underline{u} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \notin H$ because $2^2 \neq 2$

Thus H is not closed under scalar mult, so H is not a subspace of $\mathbb{R}^3$.

<u>Non-Example</u> $H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2} : a, b \in \mathbb{R} \right\}$.

is <u>NOT</u> a subspace of $\mathbb{R}^{2\times 2}$.

Every subspace must contain the zero vector, but $0_{2\times 2} \notin H$.

<u>Non-Example</u> Let $V = P_3$, the space of polynomials of deg ≤ 3

let H — — the set — — — — deg $= 3$.

Then H is not a subspace of V because it does not contain the zero polynomial

<u>Example</u> Let $V = C[-2,2] = \{f : [-2,2] \to \mathbb{R} \mid f \text{ is continuous}\}$.

Let $H = \{f \in V \mid f(1) = 0\}$.

Then H is a subspace of V.

<u>Pf</u> Note that $\underline{0} \in H$, so $H \neq \emptyset$.

Closure properties:

(i) If $f, g \in H$, then $f(1) = 0$ & $g(1) = 0$.

so $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$.

$\Rightarrow$ $f + g \in H$ and H is closed under addition.

(ii) If $\alpha \in \mathbb{R}$, $f \in H$, then $f(1) = 0$.

so $(\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0$.

$\Rightarrow$ $\alpha f \in H$ so H is closed under scalar mult.

(i) & (ii) $\Rightarrow$ H is a subspace of V, as claimed

<u>Summary</u> How to verify whether a subset H of a v.sp. V is a subspace:

- check whether $\underline{0} \in H$. If $\underline{0} \notin H$, then H is not a subspace

If $\underline{0} \in H$, at least we know $H \neq \emptyset$

- if you think H is a subspace, check:
  (i) H is closed under addition
  AND. (ii) H — — — — — scalar mult.
- if you think H is not a subspace, you need to find a counterexample to either (i) or (ii)