Appendix

Useful trigonometric formulae

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x\cos x$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B, \quad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

Some derivatives

In the table below, some derivatives are listed

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1/\cosh^2 x$
$\log x$	$\frac{1}{x}$

Useful integrals

$$\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}, \quad \forall a \neq -1; \quad \text{and} \quad \int \frac{1}{x}\,dx = \ln|x| \quad \text{for } a = -1$$

$$\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x,$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right), \quad \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{|x-a|}{|x+a|},$$

Exact first-order ODEs

If the equation

$$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0$$

is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

Reducible to separable ODEs:

$$y' = f(ax + by + c) \quad \Rightarrow \quad z = ax + by + c;$$
$$y' = f\left(\frac{y}{x}\right) \quad \Rightarrow \quad y = x\,z$$

Variation of parameter method for first-order ODEs

Given the inhomogeneous ODE

$$y' = A(x)\,y + B(x)$$

It starts with finding the general solution $y_h(x)$ of the corresponding homogeneous equation $y' = A(x)\,y$, and proceeds by determining the particular solution $y_p(x)$ given by

$$y_p(x) = e^{\int A(x)dx} \int B(x) e^{-\int A(x)dx} dx$$

Variation of parameter method for second-order ODEs with constant coefficients

Given the inhomogeneous ODE

$$ay'' + by' + c = f(x)$$

with $\lambda_1 \neq \lambda_2$ roots of the characteristic equation of the corresponding homogeneous ODE, a particular solution $y_p(x)$ is given by

$$y_p(x) = \frac{1}{a_2(\lambda_1 - \lambda_2)} \left\{ e^{\lambda_1 x} \int f(x) e^{-\lambda_1 x} dx - e^{\lambda_2 x} \int f(x) e^{-\lambda_2 x} dx \right\}.$$

Euler ODEs

Second order linear ODE of the type

$$ax^2 y'' + bxy' + cy = 0$$

Solved by putting $x = e^t$ and deriving the equations for $z = y(x(t))$.

Picard-Lindelöf Theorem.

Let $\mathcal{D}$ be the rectangular domain in the xy plane defined as $\mathcal{D} = (|x - a| \leq A,\ |y - b| \leq B)$ and suppose $f(x, y)$ is a function defined on $\mathcal{D}$ which satisfies the following conditions:

(i) $f(x, y)$ is continuous and therefore bounded in $\mathcal{D}$

(ii) the parameters A and B satisfy $A \leq B/M$ where $M = max_{\mathcal{D}}|f(x, y)|$

(iii) $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$.

Then there exists a unique solution on $\mathcal{D}$ to the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b.$$

Educated guess method:

The educated guess method is a method to find a particular solution of inhomogeneous ODEs of the type

$$ay'' + by' + cy = f(x).$$

Under the conditions in which the method can be applied, for $f(x) = p(x)e^{\alpha x}$, a particular solution exists of the form

$$y_p(x) = Q(x)e^{\alpha x};$$

for $f(x) = p(x)\cos(\alpha x)$ or $f(x) = p(x)\sin(\alpha x)$, a particular solution exists of the form

$$y_p(x) = Q(x)[A\cos(\alpha x) + B\sin(\alpha x)]$$

, where $p(x)$ and $Q(x)$ are polynomials of the same degree.

End of Appendix.