Queen Mary
University of London
Final Exam Solutions
2018

Problem 1

similar form as tutorial problems / mid-term exams with adaptations

a) *Find the solution to the Initial Value Problem of the given first-order ordinary differential equation (ODE)*

$$y' = -y/(1+x),\ y(1) = 2.$$

(6 marks)

Solution: The ODE is separable, so applying the method of separation of variables we have $\frac{dy}{y} = -\frac{1}{1+x}dx$ (2 marks). On the left-hand side, $H(y) = \int \frac{dy}{y} = \ln|y|$, $H^{-1}(u) = \pm e^u$ (1 mark) and on the right-hand side $-\int \frac{dx}{x+1} = -\ln|1+x| + C$ (1 mark). The general solution is $y(x) = H^{-1}\left(-\ln|x+1| + C\right) = \pm\frac{1}{|x+1|}e^C = \frac{D}{|x+1|}$ (1 mark). As $y(1) = D/2 = 2$, we find the solution of the initial value problem $y(x) = \frac{4}{|x+1|}$ (1 mark).

b) *Find the general solution to the homogeneous second-order linear ODE*

$$y'' + y' - 12y = 0.$$

(6 marks)

Solution: The ODE is second-order linear ODE with constant coefficients, thus we first write down the characteristic equation is $\lambda^2 + \lambda - 12 = 0$ (2 marks). This equations has two real roots $\lambda_1 = 3, \lambda_2 = -4$ (2 marks). Hence, the general solution is $y_h = C_1e^{3x} + C_2e^{-4x}$ (2 marks).

c) *Use the solution in (b) to find the general solution to the inhomogeneous second-order linear ODE*

$$y'' + y' - 12y = -3e^{-x}.$$

(8 marks)

Solution:
First, we check whether -1 is a root of the characteristic equation of the corresponding homogeneous ODE. Since the function e^{-x} is not a solution to the homogeneous equation, we may use the educated guess.
Second, we look for the particular solution of the inhomogeneous equation in the form $y_p(x) = d_0 e^{-x}$ (2 marks).
Thus we have $y_p'(x) = -d_0 e^{-x}$ (1 mark) and $y_p''(x) = d_0 e^{-x}$ (1 mark). Substituting these back into the inhomogeneous equation gives on the left-hand side $(1-1-12)d_0 e^{-x} = -3e^{-x}$ so that to match to the right-hand side we should choose $d_0 = 1/4$, hence $y_p(x) = \frac{1}{4}e^{-x}$ (2 marks). (*If the students directly write the solution for d_0, they also get the full 4 marks in this step.*)
Finally, the general solution to the inhomogeneous equation is given by

$$y_g(x) = y_h(x) + y_p(x) = c_1 e^{3x} + c_2 e^{-4x} + \frac{1}{4}e^{-x}$$

(2 marks).

Problem 2

similar form as tutorial problems with adaptations

Consider the differential equation

$$(1 - by\sin(x)) + 2\cos(x)y' = 0.$$

a) Find the value of the parameter b such that the given differential equation is exact. (8 marks)

Solution:
Denoting $P(x,y) = 1 - by\sin x$ (1 mark), $Q(x,y) = 2\cos x$ (1 mark).
The equation is exact, when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (2 mark). As $\frac{\partial P}{\partial y} = -b\sin x$, $\frac{\partial Q}{\partial x} = -2\sin x$, thus b=2 (4 mark). (*If the students directly write the solution for b, they also get the full 6 marks in this step.*)

b) For the parameter b found in (a) above, find the solution which satisfies the initial condition $y(0) = 1$. (12 marks)

Solution:
The general solution is looked for in implicit form $F(x,y) = C$ (2 marks). Here, $F = \int P(x,y)\,dx = \int(1 - 2y\sin x)\,dx = x + 2y\cos x + g(y)$ (2 marks), where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = 2\cos x + g'(y)$ (2 marks). We therefore conclude that $g'(y) = 0$ so that $g(y) = const$ (2 marks). Thus the solution in implicit form

is $x + 2y\cos x = C$, whereas the explicit form is $y = (C - x)/2\cos x$ (2 marks). As $y(0) = C/2 = 1$, we have $C = 2$ and the solution is $y = (2 - x)/2\cos x$ (2 marks).

c) Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad f(x, y) = \sqrt{3y^2 + 16}, \quad y(1) = 0\,.$$

Show that the Picard-Lindelöf Theorem ensures the uniqueness and existence of a solution to the above problem in a rectangular domain $|x - a| \le A$, $|y - b| \le B$, and specify the parameters a and b. Write down the maximal value of the width A for $B = 4$. (10 marks)

Solution:
In our case of initial conditions $a = 1$ and $b = y(1) = 0$, hence in the rectangular domain $\mathcal{D} = (|x - 1| \le A,\ |y| \le B)$ (2 marks).
The right-hand side $f(x, y)$ is continuous everywhere in $\mathcal{D}$ (2 mark), and its derivative $\frac{\partial f}{\partial y}$ satisfies

$$\left|\frac{\partial f}{\partial y}\right| = 3|y|/\sqrt{3y^2 + 16} < 1,$$

so is bounded (2 marks).
The solution to the ODE exists and is unique provided $A \le B/M$ with $M = \max_{\mathcal{D}} \sqrt{3y^2 + 16}$. Thus, for a given B we have $M = \sqrt{3B^2 + 16}$ (2 marks). This implies that the width A should satisfy

$$A \le \frac{B}{M} = \frac{B}{\sqrt{3B^2 + 16}}$$

. Using $B = 4$, the maximal value of the width is

$$A = \frac{B}{\sqrt{3B^2 + 16}} = 1/2$$

(2 marks).

Problem 3

similar form as tutorial problems with adaptations

Write down the solution to the following Boundary Value Problem (BVP) for the second order non-homogeneous differential equation

$$2x^2\frac{d^2y}{dx^2} - 4y = f(x), \quad y(2) = 0,\ y'(3) = 0$$

by using the Green's function method along the following lines:

a) *Using that the left-hand side of the ODE is in the form of an Euler-type equation determine the general solution of the corresponding homogeneous ODE.* (8 marks)

Solution: According to the general method of solving the Euler-type equation we introduce the new variable by $x = e^t$ (1 mark) and the new function $z(t)$ so that

$$z(t) = y(e^t), \quad \Rightarrow \quad \frac{dz}{dt} = e^t y', \qquad \frac{d^2z}{dt^2} = e^t y' + e^{2t} y''$$

From the above we find correspondingly that $y' = e^{-t}\dot{z}$, $y'' = e^{-2t}(\ddot{z} - \dot{z})$ (3 marks). Substituting to the Euler-type equation reduces the latter to a homogeneous equation with constant coefficients:

$$e^{2t} \cdot e^{-2t}(\ddot{z} - \dot{z}) - 2z = \ddot{z} - \dot{z} - 2z = 0,$$

(1 mark)

The corresponding characteristic equation $\lambda^2 - \lambda - 2 = 0$ has two roots: $\lambda_1 = -1$ and $\lambda_2 = 2$ and the general solution is given by:

$$z(t) = C_1 e^{-t} + C_2 e^{2t},$$

for arbitrary constants C_1 and C_2 (2 marks). Finally, substituting $t = \ln x$ gives

$$y(x) = \frac{C_1}{x} + C_2 x^2,$$

(1 mark)

b) *Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x, s)$. Further represent the solution to the BVP in terms of the found $G(x, s)$ for the particular choice $f(x) = e^x$ (you need not to evaluate the resulting integrals).* (17 marks)

Solution: The left-end boundary condition $y(2) = 0$ is imposed at $x_1 = 2$ (1 mark). By comparing it to the standard form $\alpha y'(x_1) + \beta y(x_1) = 0$ we conclude that $\alpha = 0, \beta = 1$ (1 mark). Then the left-end initial value problem for the function $y_L(x)$ is formulated as

$$y_L(x_1) = \alpha, y_L'(x_1) = -\beta, \quad \Rightarrow \quad y_L(2) = 0, y_L'(2) = -1.$$

(1 mark)

Substituting here the general solution of the homogeneous equation yields $C_1/2 + 4C_2 = 0$, $-C_1/4 + 4C_2 = -1$ so that $C_1 = 4/3$, $C_2 = -1/6$ and

$$y_L(x) = \frac{4}{3x} - \frac{x^2}{6},$$

(1 mark)

implying also $y_L'(x) = \frac{1}{3}\left(-\frac{4}{x^2} - x\right)$ (1 mark).

Similarly, $x_2 = 3$ and by comparing the right-end condition $y'(3) = 0$ (1 mark) and making it to fit the standard form $\gamma y'(x_2)+\delta y(x_2) = 0$ we should take $\gamma = 1, \delta = 0$ (1 mark). Then the right-end initial value problem for the function $y_R(x)$ is formulated as

$$y_R(x_2) = \gamma, y_R'(x_1) = -\delta, \quad \Rightarrow \quad y_R(3) = 1, y_R'(3) = 0,$$

(1 mark)
The solution of such initial value problem is given by the expression (1) with coefficients given by $C_1 = 2$, $C_2 = 1/27$ so that

$$y_R(x) = \frac{2}{x} + \frac{x^2}{27}, \quad \Rightarrow y_R'(x) = -\frac{2}{x^2} + \frac{2x}{27}$$

(2 marks).
This allows us to calculate the Wronskian $W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s)$ given by

$$W = \left(-\frac{8}{3s^3} + \frac{1}{3} + \frac{8}{27*3} - \frac{2}{6*27}s^3\right) - \left(-\frac{8}{3s^3} - \frac{4}{27*3} - \frac{2}{3} - \frac{1}{27*3}s^3\right) = \frac{31}{27}$$

(2 marks)
and as $a_2(s) = 2s^2$ we have found the functions

$$A(s) = \frac{y_R(s)}{W*a_2(s)} = \frac{27}{31*2s^2}\left(\frac{2}{s} + \frac{s^2}{27}\right) = \frac{27}{31s^3} + \frac{1}{62}$$

$$B(s) = \frac{y_L(s)}{W*a_2(s)} = \frac{27}{31*2s^2}\left(\frac{4}{3s} - \frac{s^2}{6}\right) = \frac{18}{31s^3} - \frac{9}{124}$$

(1 mark)
Finally, the Green's function for the B.V.P. is given by

$$G(x,s) = \begin{cases} \left(\frac{27}{31s^3} + \frac{1}{62}\right)\left(\frac{4}{3x} - \frac{x^2}{6}\right), & 2 \le x \le s \\ \left(\frac{18}{31s^3} - \frac{9}{124}\right)\left(\frac{2}{x} + \frac{x^2}{27}\right), & s \le x \le 3 \end{cases}$$

(2 marks)
The solution to the boundary value problem is given in terms of the Green's function by

$$y(x) = \int_2^3 G(x,s)\, f(s)\, ds = \int_2^x G(x,s)\, f(s)\, ds + \int_x^3 G(x,s)\, f(s)\, ds$$

which in our case $f(x) = e^x$ amounts to

$$y(x) = \int_2^x \left(\frac{18}{31s^3} - \frac{9}{124}\right)\left(\frac{2}{x} + \frac{x^2}{27}\right) e^s\, ds + \int_x^3 \left(\frac{27}{31s^3} + \frac{1}{62}\right)\left(\frac{4}{3x} - \frac{x^2}{6}\right) e^s\, ds$$

(2 marks) or

$$y(x) = \frac{9}{62}\left(\frac{2}{x} + \frac{x^2}{27}\right)\int_2^x \left(\frac{2}{s^3} - \frac{1}{2}\right) e^s\, ds + \frac{9}{62}\left(\frac{4}{3x} - \frac{x^2}{6}\right)\int_x^3 \left(\frac{6}{s^3} + \frac{1}{9}\right) e^s\, ds.$$

Problem 4

The coming tutorial problems will be similar as this problem and are from a resit exam three years ago

Consider a system of two nonlinear first-order ODEs:

$$\dot{x} = -x - 3y - 3x^3, \quad \dot{y} = \frac{4}{3}x - y - \frac{1}{3}x^3$$

a) *Write down in matrix form the system obtained by linearization of the above equations around the point $x = y = 0$ and find the corresponding eigenvalues and eigenvectors.* (8 points)

Solution. Discarding nonlinear terms we arrive at

$$\dot{x} = -x - 3y, \quad \dot{y} = \frac{4}{3}x - y, \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ \frac{4}{3} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(2 marks).

The characteristic equation is given by $(-1-\lambda)^2 + 4 = 0$ with two complex-conjugate roots $\lambda_{1,2} = -1 \pm 2i$ (2 marks).

The eigenvector corresponding to $\lambda_1 = -1 + 2i$ can be found from

$$\begin{pmatrix} -1 & -3 \\ \frac{4}{3} & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = (-1 + 2i) \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad \Rightarrow q_1 = -\frac{2}{3}ip_1$$

(2 marks).

so that the eigenvector can be chosen as $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix}$ (1 mark). As second eigenvector $\mathbf{u}_2$ must be the complex conjugate of $\mathbf{u}_1$ we can immediately write down $\mathbf{u}_2 = \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}$ (1 mark).

b) *Write down the general solution of the linear system. Discuss the stability of the zero solution of such a linear system and determine the value $x(t \to \infty)$.* (4 marks)

Solution. As the real part of the eigenvalues is negative the system is asymptotically stable which implies $x(t) \to 0$ as $t \to \infty$. This can be also inferred directly from the general solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{(-1+2i)t} \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix} + C_2 e^{(-1-2i)t} \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}$$

(4 marks).

c) *Find the solution of the linear system corresponding to the initial conditions* $x(0) = 2$, $y(0) = 0$. *Determine the type of equilibrium for the system and describe in words the shape of trajectory in the phase plane corresponding to the specified initial conditions. Determine the tangent vector to the trajectory at* $t = 0$. (8 marks)

Solution. From the general solution we have

$$x(t) = C_1 e^{(-1+2i)t} + C_2 e^{(-1-2i)t}, \quad \Rightarrow \quad x(0) = C_1 + C_2 = 2$$

$$y(t) = -\frac{2}{3} i \left(C_1 e^{(-1+2i)t} - C_2 e^{(-1-2i)t} \right), \quad \Rightarrow \quad y(0) = -\frac{2}{3} i (C_1 - C_2) = 0$$

which gives $C_1 = C_2 = 1$. (4 marks).
Hence the trajectory is given by coordinates

$$x(t) = e^{(-1+2i)t} + e^{(-1-2i)t} = 2e^{-t} \cos 2t$$

$$y(t) = -\frac{2}{3} i \left(e^{(-1+2i)t} - e^{(-1-2i)t} \right) = \frac{4}{3} e^{-t} \sin 2t$$

which has the shape of a spiral rotating around the origin and approaching it asymptotically for $t \to \infty$. The type of equilibrium is a stable focus . The components of the initial tangent vector determining the direction of rotation are given by $\dot{x}(0) = -2$, $\dot{y}(0) = 8/3$ (4 mark).

d) *Demonstrate how to use the function* $V(x, y) = \frac{4}{3}x^2 + 3y^2$ *to investigate the stability of the full non-linear system.* (5 marks)

Solution. $V(x, y) > 0$ for $x \neq 0, y \neq 0$ and $V(0, 0) = 0$ (2 marks). We also have the orbital derivative:

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= \frac{8}{3}x(-x - 3y - 3x^3) + 6y \left(\frac{4}{3}x - y - \frac{1}{3}x^3 \right) = -\frac{8}{3}x^2 - 6y^2 - 8x^4 - 2yx^3$$

can be larger or smaller than 0 for $(x, y) \neq (0, 0)$ (2 marks). Therefore $V(x, y)$ is not a valid Lyapunov function to ensure the stability of the solution of nonlinear equation (1 mark).