Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5123: Differential Equations

Duration: 2 hours

Date and time: 5th May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): Y. V. Fyodorov

Question 1.

(a) Find the general solution of the homogeneous ODE

$$4y'' + 4y' + y = 0.$$

[6]

(b) Find the general solution of the non-homogeneous ODE

$$4y'' + 4y' + y = \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right).$$

[12]

(c) Solve the following initial value problem

$$y' = \frac{y}{x} + x\,, \quad y(1) = 2\,.$$

[7]

Question 2.

(a) **(i)** Find all functions $f(y)$ for which the following differential equation becomes exact:

$$\frac{dy}{dx} = -\frac{x^3 + f(y)}{6xy^2 + 5y^4}, \qquad (1)$$

[5]

(ii) Suppose, $f(y)$ is chosen so that the equation (1) is exact and $f(1) = 0$. Solve (1) in implicit form. **[10]**

(b) Consider the initial value problem (IVP)

$$\frac{dy}{dx} = 2x\sqrt{|y-1|} \equiv \begin{cases} x\sqrt{y-1}, & y \geqslant 1 \\ x\sqrt{1-y}, & y < 1 \end{cases}, \quad y(0) = b\,.$$

where b is a real parameter.
Find the value of the parameter b such that the corresponding IVP may have more than one solution and explain your choice. Confirm your choice by giving at least two different solutions of the IVP in the domain $y \geqslant 1$ for such a value of the parameter. **[6]**

(c) Consider the boundary value problem (BVP)

$$y'' + b^2 y = 5 \quad , \quad y(0) = 1,\, y'\left(\frac{\pi}{2}\right) = 1\,.$$

where $b > 0$ is a real parameter. Find all positive values of the parameter b such that the corresponding BVP may have either no solution or infinitely many solutions. **[4]**

Question 3.

Write down the solution to the following Boundary Value Problem (BVP) for the second order non-homogeneous differential equation

$$\frac{1}{(x+1)}\frac{d^2y}{dx^2} - \frac{1}{(x+1)^2}\frac{dy}{dx} = f(x), \quad y(0) = 0,\ y'(1) = 0$$

by using the Green's function method along the following lines:

(a) Show that the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$ and use this fact to determine the general solution of the associated homogeneous ODE.

[4]

(b) Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x, s)$.

[14]

(c) Write down the solution to the BVP in terms of $G(x, s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = 2x$.

[7]

Question 4.

(a) Consider a system of two linear first-order ODE:

$$\dot{x} = 2x - 4y, \quad \dot{y} = ax - 6y$$

where $-\infty < a < \infty$ is a real parameter.

(i) For the particular value $a = -5$ determine eigenvalues and eigenvectors associated with the system, find equations for stable and unstable invariant manifolds and sketch the phase portrait.

[**11**]

(ii) Classify for which values of the parameter a the equilibrium point $x = y = 0$ of the system represents (i) a focus, (ii) a node, and (iii) a saddle. For which values of the parameter a is the equilibrium not hyperbolic?

[**9**]

(b) Demonstrate how to use the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ for investigating the global stability of the following system of two nonlinear first-order ODEs:

$$\dot{x} = -x^3 + 2y^3, \quad \dot{y} = -2xy^2 \, .$$

[**5**]

End of Paper—An appendix of 2 pages follows.

Useful Facts.

- **Useful integrals**:

$$\int x^a\,dx = \frac{1}{a+1}\,x^{a+1}, \quad \forall a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| \quad \text{for } a=-1; \quad \int \ln x\,dx = x\ln|x| - x$$

$$\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x,$$

$$\int \sin x\cos x\,dx = \frac{1}{2}\sin^2 x, \quad \int \tan x\,dx = -\ln|\cos x|$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right), \quad a \neq \pm ib$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right), \quad a \neq \pm ib$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{|x-a|}{|x+a|},$$

- **Useful trigonometric formulae**:

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x\cos x$$

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B, \quad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$

- **Reminder on ODEs:**

If the equation $P(x,y)+Q(x,y)\,\dfrac{dy}{dx} = 0$ is exact, its solution can be found in the form $F(x,y) = Const.$ where

$$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}$$

- If there exists a unique solution $y(x)$ to a non-homogeneous **boundary value** problem for ODE $\mathcal{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x) = f(x)$ in an interval $x \in [x_1, x_2]$ with linear homogeneous B.C.

$$\alpha y'(x_1) + \beta y(x_1) = 0, \quad \gamma y'(x_2) + \delta y(x_2) = 0$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s) = \begin{cases} A(s)\, y_L(x), & x_1 \leqslant x \leqslant s \\ B(s)\, y_R(x), & s \leqslant x \leqslant x_2 \end{cases}$$

where

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems:

$$\mathcal{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta; \quad \text{and} \quad \mathcal{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta$$

End of Appendix.