Queen Mary
University of London

Main Examination period 2017

MTH5123: Differential Equations

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: R. Klages, S. Beheshti

Question 1. [25 marks]

(a) Find the general solution of the homogeneous ordinary differential equation (ODE)

$$y'' + 2y' - 15y = 0\,.$$ **[5]**

(b) Find the general solution of the inhomogeneous ODE

$$y'' + 2y' - 15y = -4e^{x}\,.$$ **[11]**

(c) Find the general solution of the first order homogeneous linear ODE

$$y' = \tan(x)\,y\,.$$ **[5]**

(d) Use the solution in c) to solve the initial value problem for the first order linear inhomogeneous ODE

$$y' = \tan(x)\,y + \sin x\,,\; -\pi/2 < x < \pi/2\,,\; y(0) = 1$$

by the variation of parameters method. **[4]**

Question 2. [25 marks]

(a) Find all functions $f(y)$ such that the following differential equation becomes exact:

$$x^2 + \frac{f(y)}{x} + \ln(xy)\,\frac{dy}{dx} = 0 \quad , \quad x > 0\,, y > 0\,.$$ **[5]**

(b) Solve the equation in (a) in implicit form for a particular choice of $f(y)$ ensuring exactness such that $f(0) = 0$. **[11]**

(c) Consider the initial value problem

$$\frac{dy}{dx} = f(x,y)\,,\; f(x,y) = \sqrt{25 + 4y^2}\,,\; y(1) = 0\,.$$

Show that the Picard-Lindelöf Theorem guarantees the existence and uniqueness of the solution of the above problem in a rectangular domain $\mathscr{D} = (|x-a| \leq A\,, |y-b| \leq B)$ in the xy plane, and specify the parameters a and b. Find the possible range of values of the height B of the domain $\mathscr{D}$ given that the width A of the domain satisfies $A < 1/3$. **[9]**

Question 3. [25 marks] Find the solution of the following boundary value problem (BVP) for the second order inhomogeneous ODE

$$\frac{1}{\cos x}\frac{d^2y}{dx^2}+\left(\frac{\sin x}{\cos^2 x}\right)\frac{dy}{dx}=f(x)\,,\, y(0)=0\,,\, y\left(\frac{\pi}{4}\right)=0$$

by using the Green's function method along the following lines:

(a) Show that the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$. Use this fact to determine the general solution of the associated homogeneous ODE. **[4]**

(b) Formulate the left-end and right-end initial value problems corresponding to the above BVP. **[9]**

(c) Use the solutions of these initial value problems to construct the Green's function $G(x,s)$ of the BVP. **[5]**

(d) Write down the solution of the BVP in terms of $G(x,s)$ and $f(x)$. Use it to find the explicit form of the solution for $f(x)=1$. **[7]**

Question 4. [25 marks]
Consider the system of two nonlinear first-order ODEs

$$\dot{x}=-4y-x^3\,,\,\dot{y}=3x-y^3\,. \tag{1}$$

(a) Write down in matrix form the linear system obtained by linearization of the above equations around the fixed point $x=y=0$. Then find the corresponding eigenvalues and eigenvectors. **[8]**

(b) Determine the type of fixed point for the linear system. Is it stable? Is it asymptotically stable? Can one judge the stability of the nonlinear system by the linear approximation? **[4]**

(c) Write down the general solution of the linear system. **[2]**

(d) Find the solution of the linear system for the initial conditions $x(0)=2$, $y(0)=0$ in terms of real-valued functions. What is the shape of the corresponding trajectory in the phase plane? **[6]**

(e) Demonstrate how to use the function $V(x,y)=3x^2+4y^2$ to investigate the stability of the original nonlinear system (1). **[5]**

End of Paper—An appendix of 2 pages follows.

Formula Sheet

- **Useful integrals:**

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1}, \quad \forall a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| \quad \text{for } a=-1, \quad \int \ln x\,dx = x\ln|x| - x$$

$$\int \cos x\,dx = \sin x, \quad \int \sin x\,dx = -\cos x$$

$$\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x, \quad \int \tan x\,dx = -\ln|\cos x|$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right), \quad a \neq \pm ib$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right), \quad a \neq \pm ib$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{|x-a|}{|x+a|}$$

- **Useful trigonometric formulae:**

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2\sin x\cos x$$

$$\sin(A\pm B) = \sin A\cos B \pm \cos A \sin B, \quad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

- **Reminder on solving ODEs:**
  - The ODE
  $$y' = A(x)\,y + B(x)$$
  is solved by the variation of parameters method: One starts with finding the solution of the corresponding homogeneous equation $y' = A(x)\,y$. One then proceeds by replacing the constant of integration with a function to be determined.
  - If the ODE
  $$P(x,y) + Q(x,y)\,\frac{dy}{dx} = 0$$
  is *exact*, its solution can be found in the form $F(x,y) = Const.$, where
  $$P = \frac{\partial F}{\partial x} \quad \text{and} \quad Q = \frac{\partial F}{\partial y}\,.$$

- For the **initial value problem**

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b$$

the Picard-Lindelöf Theorem guarantees the existence and uniqueness of the solution in a rectangular domain $\mathscr{D} = (|x-a| \leq A\,, |y-b| \leq B)$ centered at the point (a,b) in the xy plane provided the following conditions are satisfied:
(**i**) $f(x,y)$ is continuous and therefore bounded in $\mathscr{D}$
(**ii**) the partial derivative $|\frac{\partial f}{\partial y}|$ is bounded in $\mathscr{D}$
(**iii**) the parameters A and B satisfy $A < B/M$, where $M = max_{\mathscr{D}}|f(x,y)|$.

- If there exists a unique solution $y(x)$ to an inhomogeneous **boundary value problem** for ODE $\mathscr{L}(y) = a_2(x)y'' + a_1(x)y' + a_0(x) = f(x)$ in an interval $x \in [x_1, x_2]$ with linear homogeneous boundary condition

$$\alpha y'(x_1) + \beta y(x_1) = 0, \quad \gamma y'(x_2) + \delta y(x_2) = 0$$

it can be found by the Green's function method:

$$y(x) = \int_{x_1}^{x_2} G(x,s)\, f(s)\, ds, \quad G(x,s) = \begin{cases} A(s)\, y_L(x), & x_1 \leq x \leq s \\ B(s)\, y_R(x), & s \leq x \leq x_2 \end{cases}$$

where

$$A(s) = \frac{y_R(s)}{a_2(s)W(s)}, \quad B(s) = \frac{y_L(s)}{a_2(s)W(s)}, \quad W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s)$$

and $y_L(x), y_R(x)$ are solutions to the left/right initial value problems

$$\mathscr{L}(y) = 0,\ y(x_1) = \alpha,\ y'(x_1) = -\beta \quad \text{and} \quad \mathscr{L}(y) = 0,\ y(x_2) = \gamma,\ y'(x_2) = -\delta\,.$$

- The orbital derivative for a Lyapunov function $V(x,y)$ is defined as

$$\mathscr{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}\,.$$

End of Appendix.