May MTH5123 Exam

1. **Question 1a**

   a) Which of the following ordinary differential equations (ODEs) has t as the independent variable and can be solved by the separation of variables method? (4 marks)

   • I ✓ • II ✓ • III • IV

   where I: $\frac{dy}{dt} = y^2 \tanh(t^2 + 4t)$, II: $\dot{y} = e^y e^t + e^t$, III: $y' = 2xy^2$, IV: $\dot{y} = \sin y + e^t$

   b) Which of the following ODEs can be reduced to be separable? (4 marks)

   • I ✓ • II • III ✓

   where I: $y' = \ln(y) - \ln(x) - 3$, II: $y' = \tanh((y/x)^2) + 2e^{x+y}$, III: $\dot{y} = ty$.

   c) Which of the following statements is correct for solving the ODE, $\dot{x} = 3\cos(t/x) + 9(x/t)^2$? (4 marks)

   • The ODE can be reduced to be separable;
   • This is an inhomogeneous linear ODE that can be solved by the variation of parameter method;
   • The method to solve exact ODEs can be applied;

   d) The solution to the initial value problem $y' = \frac{4}{\cos(y)}, y(0) = \pi/2$ is, (8 marks)

   • I • II • III • IV

   where
   I:$y(x) = \arcsin(C + 4x)$, where C is an arbitrary constant,
   II:$y(x) = \arcsin(4x + 1)$
   III:$y(x) = 4\arccos(-1 + 4x)$
   IV:$y(x) = \frac{1}{4}\sin(x + 1)$

2. **Question 1b**

   a) Which of the following ordinary differential equations (ODEs) has

t as the independent variable and can be solved by the separation of variables method? (4 marks)

• I • II • III ✓ • IV ✓

where I: $\frac{dy}{dx} = e^y \sin(y)$, II: $\dot{y} = t + e^{t+y}$, III: $\dot{y} = t^2 \cos(y+5)$, IV: $\dot{y} = e^{t^2+y}$

b) Which of the following ODEs can be reduced to be separable? (4 marks)

• I ✓ • II ✓ • III

where I: $y' = 3\ln(y) - 3\ln(x) + 10\frac{y}{x}$, II: $y' = \tanh((y/x)^2) + 2e^{x/y}$, III: $\dot{y} = (t+3y)t$.

c) Which of the following statements is correct for solving the ODE, $\dot{x} = 3\cos(5x + 7t - 2)$? (4 marks)

• The ODE can be reduced to be separable;
• This is an inhomogeneous linear ODE that can be solved by the variation of parameter method;
• The method to solve exact ODEs can be applied;

d) The solution to the initial value problem $y' = \frac{3}{\sin(y)}, y(0) = \pi/2$ is, (8 marks)

• I • II • III • IV

where
I:$y(x) = \arcsin(C + 3x)$, where C is an arbitrary constant,
II:$y(x) = \arcsin(C - 3x)$, where C is an arbitrary constant,
III:$y(x) = \arccos(3x)$
IV:$y(x) = \arccos(-3x)$

3. **Question 2a**

   Find the right match for the following ODEs in the dropdown menu

   (a) $y' = \sin(x/y)$ 1st-order reducible to separable ODE
   (b) $y'' = 5x + 3y + 2$ None of the above forms
   (c) $\dot{y} = e^t + t^2 y$ 1st-order linear ODE
   (d) $3xy' = y - 2x^2 y''$ Euler-type ODE

(e) $y = y'' \sin(x) + e^x + 2$ 2nd-order linear inhomogeneous ODE

4. **Question 2b**

   Find the right match for the following ODEs in the dropdown menu

   (a) $y'' - \sin(x) = y \ln(x)$ 2nd-order linear inhomogeneous ODE
   (b) $\dot{y} = y \sin(t^2) + 6$ 1st-order linear ODE
   (c) $y'' = 3x - 2y + 6$ None of the above forms
   (d) $y' = \tanh(x + y)$ 1st-order reducible to separable ODE
   (e) $9xy' = y - 7x^2 y''$ Euler-type ODE

5. **Question 3a**

   a) Consider the initial value problem (IVP) $y' = yx/(x^2 - 1), \quad y(0) = 1$. Does this IVP satisfy the hypotheses of the Picard-Lindelöf theorem? (5 marks)

   - Yes because the function $f(x, y) = yx/(x^2 - 1)$ is continuous in a sufficiently small rectangular region D centered in the point of the xy plane of coordinates$(x_0, y_0) = (0, 1)$.
   - No because the function $f(x, y) = yx/(x^2 - 1)$ is not continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates$(x_0, y_0) = (0, 1)$.
   - Yes because the function $f(x, y) = yx/(x^2 - 1)$ and its partial derivative $\partial f(x, y)/\partial y$ are continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (0, 1)$.
   - No because the neither the function $f(x, y) = yx/(x^2 - 1)$ nor its partial derivative $\partial f(x, y)/\partial y$ are continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (0, 1)$.

   b) What is the solution to the initial value problem in point (a) valid sufficiently close to the initial condition? (5 marks)

   - I
   - II
   - III ✓

where I:$y(x) = 2\sqrt{|x^2-1|}$, II:$y(x) = \sqrt{x^2-1}$, III:$y(x) = \sqrt{1-x^2}$.

c) Consider the initial value problem (IVP) $y' = yx/(x^2-1), \quad y(1) = 0$. Does this IVP satisfy the hypotheses of the Picard-Lindelöf theorem? (5 marks)

- Yes because the function $f(x,y) = yx/(x^2-1)$ is continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates$(x_0, y_0) = (1, 0)$.
- No because the function $f(x,y) = yx/(x^2-1)$ is not continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (1, 0)$. ✓
- Yes because the function $f(x,y) = yx/(x^2-1)$ and its partial derivative $\partial f(x,y)/\partial y$ are continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (1, 0)$.

d) How many solutions does the IVP in point (d) have? (5 marks)

- None
- 1
- 2
- Infinite ✓

6. **Question 3b**

a) Consider the initial value problem (IVP) $y' = 2yx/(x^2-4), \quad y(0) = 1$. Does this IVP satisfy the hypotheses of the Picard-Lindelöf theorem? (5 marks)

- Yes because the function $f(x,y) = 2yx/(x^2-4)$ is continuous in a sufficiently small rectangular region D centered in the point of the xy plane of coordinates$(x_0, y_0) = (0, 1)$.
- No because the function $f(x,y) = 2yx/(x^2-4)$ is not continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates$(x_0, y_0) = (0, 1)$.
- Yes because the function $f(x,y) = 2yx/(x^2-4)$ and its partial derivative $\partial f(x,y)/\partial y$ are continuous in a sufficiently small rectan-

gular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (0, 1)$.

- No because the neither the function $f(x, y) = 2yx/(x^2 - 4)$ nor its partial derivative $\partial f(x, y)/\partial y$ are continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (0, 1)$.

b) What is the solution to the initial value problem in point (a) valid sufficiently close to the initial condition? (5 marks)

- I
- II
- III ✓

where I:$y(x) = 1 - x^2/4$, II:$y(x) = 1 - x^2$, III:$y(x) = \sqrt{4 - x^2}/2$.

c) Consider the initial value problem (IVP) $y' = 2yx/(x^2-4), \quad y(-2) = 0$. Does this IVP satisfy the hypotheses of the Picard-Lindelöf theorem? (5 marks)

- Yes because the function $f(x, y) = 2yx/(x^2 - 4)$ is continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates$(x_0, y_0) = (-2, 0)$.
- No because the function $f(x, y) = 2yx/(x^2 - 4)$ is not continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (-2, 0)$. ✓
- Yes because the function $f(x, y) = 2yx/(x^2 - 4)$ and its partial derivative $\partial f(x, y)/\partial y$ are continuous in a sufficiently small rectangular region D centred in the point of the xy plane of coordinates $(x_0, y_0) = (-2, 0)$.

d) How many solutions does the IVP in point (c) have? (5 marks)

- None
- 1
- 2
- Infinite ✓

7. **Question 4a**

(a) Consider the boundary value problem (BVP) $3y'' + 12y = \sin(2x)$ $y(0) = 0, y(\pi/2) = 1$. Does this BVP have a unique solution ? (5 marks)

- Yes
- No ✓
- It is impossible to determine the ODE cannot be solved.

(b) For which real value of b the following BVP $3y'' = -12y + \sin(b)$ $y(0) = 0, y(\pi/2) = 1$ has a unique solution ? (5 marks)

• $b = n\pi$ with n integer • $b \neq n\pi$ with n integer • Any value of b • No value of b ✓

8. **Question 4b**

(a) Consider the boundary value problem (BVP) $5y''+45y = -\tan(7x)$ $y(0) = 0, y(\pi/7) = 1$. Does this BVP have a unique solution ? (5 marks)

- Yes ✓
- No
- It is impossible to determine the ODE cannot be solved.

(b) For which real value of b the following BVP $5y'' = -45y + \cos(3b)$ $y(0) = 0, y(\pi/3) = -5$ has a unique solution ? (5 marks)

• $b = n\pi/2$ with n integer • $b \neq n\pi/2$ with n integer • Any value of b • No value of b ✓

9. **Question 5a**

(a) Consider a system of two ordinary differential equations: $\dot{y}_1 = \tan(\frac{1}{2}y_1 - y_2) - y_2^2, \quad \dot{y}_2 = \sin(y_1) + \frac{1}{2}\sin(y_2)$.
Linearise the system of ODE close to the $(y_1, y_2) = (0, 0)$ equilibria. The phase portrait of the linearised system displays a: (5 marks)

• Stable node • Unstable node • Saddle • Unstable focus with spiral out ✓ • Centre • Stable focus with spiral in

(b) For which real value of a the system of ODEs
$\dot{y}_1 = \sin(ay_2 + 2y_1), \quad \dot{y}_2 = -\tanh(y_1 + ay_2)$ when linearised displays a saddle at $(y_1, y_2) = (0, 0)$? (5 marks)

- $a < 2$
- $a > 0$ ✓
- $a < -2$
- $-2 < a < 0$

10. **Question 5b**

(a) Consider a system of two first-order ordinary differential equations:
$\dot{y}_1 = 2e^{y_1} - 2 + ay_2, \quad \dot{y}_2 = -2\tanh(y_1 + y_2 + y_2^3)$.
Linearise the system of ODE close to the $(y_1, y_2) = (0, 0)$ equilibria. For $a = 1$ the phase portrait of the linearised system is: (5 marks)

• Stable node • Unstable node • Saddle ✓ • Unstable focus with spiral out • Centre • Stable focus with spiral in

(b) For which real value of a the system of ODEs
$\dot{y}_1 = a\sin(y_1 + y_2) + \tanh(y_1^3), \quad \dot{y}_2 = -\tanh(y_1 + y_1^2) - ay_2$ when linearised, displays a centre around the equilibria $(y_1, y_2) = (0, 0)$? (5 marks)

- Any value of a
- $a > 0$
- $-1 < a < 0$ ✓
- $a < -1$

11. **Question 6**

This question requires an handwritten answer which should be uploaded here in a the format of a single combined pdf.

a) Consider the ODE describing the motion of a pendulum in presence of friction. Let θ indicate the angle of the pendulum with respect to the vertical line and let t indicate the time.
The ODE describing the motion of the pendulum is given by

$$m\ell\ddot{\theta} = -mg\sin\theta - \gamma\dot{\theta}, \tag{1}$$

with $\theta \in [-\pi/2, \pi/2]$. Here $m > 0$ indicates the mass of the pendulum, $\ell > 0$ indicates its length, $g > 0$ indicates the gravitational constant, and $\gamma \leq 0$ is a constant real parameter indicating the intensity of the friction.

- Identify the dependent and independent variable in this ODE. (1 marks)
- Is this a linear or non-linear ODE? (2 mark)
- Which is the order of this ODE? (2 mark)

b) Consider the ODE introduced in point (a) and describing the motion of the pendulum

$$m\ell\ddot{\theta} = -mg\sin(\theta) - \gamma\dot{\theta}. \tag{2}$$

with $\theta \in [-\pi/2, \pi/2]$. Put $m = 1$ and $\ell = 1$.

- Convert this ODE into a system of two first-order ODEs. (4 marks)
- Compute all equilibria of this system of ODEs.
  Linearise this system of ODE around each equilibria.
  Find the eigenvalues of the linearised system around each equilibria. (9 marks)
- Assume that $g > 0$ is constant but that $\gamma \geq 0$ can be tuned.
  For which values of γ the phase portraits of the linearised systems are fixed points?
  For which values of γ the phase portraits of the linearised systems are stable focuses?
  For which values of γ the phase portraits of the linearised systems are centres? (9 marks)
- Explain the meaning of your results. (3 marks)