Queen Mary University of London

MTH5123 Differential Equations

Formative Assessment: Week 10 G. Bianconi

- Each Coursework consists of three parts:
  - **I.** Practice problems
  - **II.** Mock Quiz
  - **III.** Exploration problems
- A selection of solutions to the listed problems will be posted on QMPlus by the end of Week 10 and discussed during the tutorials.
- I encourage all students to learn and check their computational answers using math software such as Mathematica, MATLAB, etc. Using numerical software is a fun practice and will help you to visualise your solutions.

I. Practice Problems

A. Determine the type of equilibrium at $y_1 = 0, y_2 = 0$ for the following ODE systems . *Hint:* Both question appeared in Coursework 8, and you can check the solutions to the IVP and also sketch of trajectories there.

1) $\dot{y_1} = -\frac{1}{2}y_1 + \frac{5}{2}y_2$, $\dot{y_2} = \frac{5}{2}y_1 - \frac{1}{2}y_2$, $y_1(0) = a, y_2(0) = b$.

2) $\dot{y_1} = -y_1 + 5y_2$, $\dot{y_2} = -y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 4$.

B. Determine the general solution and sketch the phase portraits of the following systems of linear differential equations:

1) $\dot{y_1} = -y_1 + 6y_2$, $\dot{y_2} = -3y_1 + 8y_2$

2) $\dot{y_1} = -y_1 + y_2$, $\dot{y_2} = y_1 - y_2$

3) $\dot{y_1} = -4y_1 - 8y_2$, $\dot{y_2} = 4y_1 + 4y_2$

C. Determine the type of fixed point for the dynamical systems

$$\dot{y_1} = 4y_2 \ , \ \dot{y_2} = -y_1 .$$

Then determine the solutions of the corresponding initial value problems for the general initial conditions $y_1(0) = a, y_2(0) = b$. Sketch the phase portraits in the (y_1, y_2) phase plane.

D. Determine the solution of the initial value problem

$$\dot{y_1} = y_1 - 4y_2 \ , \ \dot{y_2} = 4y_1 + y_2 \ , \ y_1(0) = 0 \ , \ y_2(0) = 1 \ , \ t \geq 0$$

and the type of fixed point. Then sketch the trajectory in the (y_1, y_2) phase plane corresponding to the chosen initial values in the specified range of t.

II. Mock Quiz

Train yourself for Coursework 2 by answering Mock Quiz Week 10.

III. Further Exploration: Applications involving Dynamical Systems

Second order, constant-coefficient linear differential equations appear in many physical models, such as the spring-mass system studied in the first half of the semester. In this exercise, we see a second example, using electrical circuits.

To model the flow of electric current in a simple series circuit (involving a resistor a capacitor and an inductor in series), one uses Kirchhoff's Law to obtain

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t).$$

Here L, R, and C are not variables but positive parameters (referred to as the inductance, resistance and capacitance, respectively). $E(t)$ is the impressed voltage (in volts) which is a function of our independent variable t. Q and I are both variables depending on t, where Q is the total charge on the capacitor at time t (in coulombs) and I is the current at time t (in amperes). In addition, the relation between charge and current is $I = dQ/dt$.

A. 1) Rewrite the above equation as a 2nd-order ODE in the charge Q (i.e. an ODE only containing variable Q, independent variable t, functions of t, and parameters L, R, and C.
2) Show how to get a new 2nd-order ODE in the current I as

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \dot{E}(t).$$

B. 1) Assuming $\dot{E}(t) = 0$ (which means the system is closed), write the above 2nd-order ODE in I as a system of two 1st-order ODEs with variables y_1 and y_2. (Hint: assume $y_1 = I$ and $y_2 = dI/dt$ and follow the method we learned in 2.1 scanned lecture notes).
2) Show that $y_1 = 0$, $y_2 = 0$ is a critical point.

In the next Coursework, we shall analyze the nature and stability of the critical point as a function of the parameters (in this case, L, R and C).

Remark: exactly the same analysis can be used to study the equation of motion for a damped spring-mass system $m\ddot{u} + c\dot{u} + ku = 0$, where m, k, c are positive constants. Full details for the derivation of the equation appearing in this example can be found in Boyce & DiPrima, Section 3.7.