MTH5123 Differential Equations

Formative Assessment Week 6 – Selected Solutions G. Bianconi

I. Practice Problems

A. Find the solution to the following BVP for the given ODE

$$x^2 \frac{d^2y}{dx^2} - 2y = 0, \quad y(1) = 0, \; y'(1) = 1.$$

Solution: According to the general method of solving the Euler-type equation we introduce the new variable by $x = e^t$ and the new function $z(t)$ so that

$$z(t) = y(e^t), \quad \Rightarrow \quad \frac{dz}{dt} = e^t y', \qquad \frac{d^2z}{dt^2} = e^t y' + e^{2t} y''$$

From the above we find correspondingly that $y' = e^{-t}\dot{z}$, $y'' = e^{-2t}(\ddot{z} - \dot{z})$. Substituting to the Euler-type equation reduces the latter to a homogeneous equation with constant coefficients:

$$e^{2t} \cdot e^{-2t}(\ddot{z} - \dot{z}) - 2z = \ddot{z} - \dot{z} - 2z = 0.$$

The corresponding characteristic equation $\lambda^2 - \lambda - 2 = 0$ has two roots: $\lambda_1 = -1$ and $\lambda_2 = 2$ and the general solution is given by:

$$z(t) = C_1 e^{-t} + C_2 e^{2t},$$

for arbitrary constants C_1 and C_2. Finally, substituting $t = \ln x$ gives $y_h(x) = \frac{C_1}{x} + C_2 x^2$. As the initial conditions include the derivative $y'(x)$ at $x = 1$, thus we first differentiate our general solution and have $y_h'(x) = -\frac{C_1}{x^2} + 2C_2 x$. Using the initial conditions, we have $y(1) = \frac{C_1}{1} + C_2 = 0$, and $y'(1) = -C_1 + 2C_2 = 1$. Thus, $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$, and the solution to this BVP is $y(x) = -\frac{1}{3x} + \frac{x^2}{3}$.

B. Consider the following boundary value problem (BVP)

$$\frac{1}{\cos x} \frac{d^2y}{dx^2} + \left(\frac{\sin x}{\cos^2 x} \right) \frac{dy}{dx} = 0 \;, \; y(0) = 0 \;, \; y\left(\frac{\pi}{4}\right) = 2$$

Show that the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$. Use this fact to determine the solution to the above BVP.

Solution: We have

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) = r(x)\frac{d^2y}{dx^2} + r'(x)\frac{dy}{dx} \, ,$$

which coincides with the original ODE for $r(x) = \frac{1}{\cos x}$. Therefore, the homogeneous ODE has the form

$$\frac{d}{dx}\left(\frac{1}{\cos x}\frac{dy}{dx}\right) = 0 \, .$$

This can be integrated to find the general solution

$$\frac{1}{\cos x}\frac{dy}{dx} = C_1 \quad \Rightarrow \quad y(x) = C_1 \sin x + C_2$$

for real constants C_1 and C_2. Using the initial conditions, we have $y(0) = C_1 \sin 0 + C_2 = C_2 = 0$, and $y\left(\frac{\pi}{4}\right) = C_1 \sin\frac{\pi}{4} + C_2 = C_1\frac{\sqrt{2}}{2} + C_2 = \frac{\sqrt{2}}{2}C_1 = 2$. Thus, the solution to this BVP is $y(x) = 2\sqrt{2}\sin x$.

C. Find the solution to the following Boundary Value Problem for the second order inhomogeneous differential equation

$$\frac{d^2y}{dx^2} = x \, , \; y(-1) = 0 \, , \; y(1) = 0.$$

Solution: The general solution $y_h(x)$ to the linear homogeneous equation $y'' = 0$ is found through the characteristic equations $\lambda^2 = 0$. Thus, there are two identical real roots $\lambda_1 = \lambda_2 = \lambda = 0$. Accordingly, the general solution to the homogenous ODE is given by

$$y_h(x) = (c_1 x + c_2)\mathrm{e}^{\lambda x} = c_1 x + c_2 \, .$$

Now we need to find the solution for the general solution to the inhomogeneous ODE.

Note: 1. we can not directly use the equation for the particular solution (derived by the variation of parameter method) as
$y_p(x) = \frac{1}{(\lambda_1-\lambda_2)a_2}\left\{e^{\lambda_1 x}\int f(x)e^{-\lambda_1 x}dx - e^{\lambda_2 x}\int f(x)e^{-\lambda_2 x}dx\right\}$, because this equation is obtained under the assumption of two distinct roots (real or complex), $\lambda_1 - \lambda_2 \neq 0$.
2. we also can not use the educated guess method as introduced in our lectures, because the right hand side of the ODE is x, which can be written as $x\mathrm{e}^{0x}$ and 0 is the root of the characteristic equation for the homogenous ODE.

Instead, we can use the variation of parameter method directly. Based on the previous result that $y_h(x) = c_1 x + c_2$, we assume the general solution has the form as $y_g(x) = c_1(x)x + c_2(x)$, where $c_1(x)$ and $c_2(x)$ are unknown and need to be determined. Thus, $y_g'(x) = c_1'(x)x + c_1(x) + c_2'(x)$. Assuming $c_1'(x)x + c_2'(x) = 0$ (!!!! important trick, see lecture notes for details), we have $y_g'(x) = c_1'(x)x + c_1(x) + c_2'(x) = c_1(x)$ and thus $y_g''(x) = c_1'(x)$.

Putting $y_g''(x)$ back to the inhomogeneous ODE $\frac{d^2y}{dx^2} = x$, we have $y_g''(x) = c_1'(x) = x$. Thus, we obtain $c_1(x) = \frac{1}{2}x^2 + D_1$, where D_1 is an arbitrary real constant. Using $c_1'(x) = x$ in the assumption $c_1'(x)x + c_2'(x) = 0$, we have $c_2'(x) = -x^2$ and $c_2(x) = -\frac{1}{3}x^3 + D_2$, where D_2 is an arbitrary real constant. Up so far, we determined $c_1(x)$ and $c_2(x)$, and the general solution to the inhomogeneous ODE is $y_g(x) = c_1(x)x + c_2(x) = (\frac{1}{2}x^2 + D_1)x - \frac{1}{3}x^3 + D_2 = \frac{1}{6}x^3 + D_1 x + D_2$.

Finally, using the initial conditions, $y(-1) = -\frac{1}{6} - D_1 + D_2 = 0$ and $y(1) = \frac{1}{6} + D_1 + D_2 = 0$, we have $D_1 = -\frac{1}{6}$ and $D_2 = 0$, which yields the solution to this BVP as

$$y(x) = \frac{1}{6}(x^3 - x)\,.$$

D. Find the solution of the following Boundary Value Problem for the second order linear inhomogeneous differential equation,

$$(x+1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x)\,, f(x) = -1,\ y(0) = 0\,,\ y'(1) = 0.$$

Hint: the left-hand side of the ODE can be written down in the form $\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right)$ for some function $r(x)$ and use this fact to determine the general solution of the associated homogeneous ODE $y_h(x)$. Based on $y_h(x)$, using the variation of parameter method to find the general solution to the inhomogeneous ODE $y_g(x)$. Useful formula: $\int \ln z dz = z(lnz - 1) + c$.

Solution: We have

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) = r(x)\frac{d^2y}{dx^2} + r'(x)\frac{dy}{dx}.$$

which coincides with the original ODE for $r(x) = x + 1$. The homogeneous ODE has therefore the form

$$\frac{d}{dx}\left((x+1)\frac{dy}{dx}\right) = 0$$

and can be integrated to find the general solution

$$(x+1)\frac{dy}{dx} = c_1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{c_1}{x+1} \quad \Rightarrow \quad y_h(x) = c_1 \ln|x+1| + c_2$$

for constants c_1 and c_2. Because we search the solution to our original BVP in the interval of $x \in [0, 1]$ according to the BCs, thus we can use $0 \le x \le 1$ we can omit the modulus sign and write simply $y_h(x) = c_1 \ln(x+1) + c_2$, where $x \in [0, 1]$.

Based on the previous result that $y_h(x) = c_1 \ln(x+1) + c_2$, we assume the general solution has the form as $y_g(x) = c_1(x)\ln(x+1) + c_2(x)$, where $c_1(x)$ and $c_2(x)$ are unknown and need to be determined. Thus, $y_g'(x) = c_1'(x)\ln(x+1) + \frac{1}{x+1}c_1(x) + c_2'(x)$. Assuming $c_1'(x)\ln(x+1) + c_2'(x) = 0$ (!!!! important trick, see lecture notes for details), we have $y_g'(x) = c_1'(x)\ln(x+1) + \frac{1}{x+1}c_1(x) + c_2'(x) = \frac{1}{x+1}c_1(x)$ and thus $y_g''(x) = \frac{c_1'(x) - c_1(x)}{(1+x)^2}$.

Putting $y_g(x) = c_1(x)\ln(x+1) + c_2(x)$,$y_g'(x) = \frac{1}{x+1}c_1(x)$, $y_g''(x) = \frac{c_1'(x)(x+1)-c_1(x)}{(1+x)^2}$ back to the inhomogeneous ODE $(x+1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = -1$, we have $c_1'(x) = -1$. Thus, we obtain $c_1(x) = -x + D_1$, where D_1 is an arbitrary real constant. Using $c_1'(x) = -1$ in the assumption $c_1'(x)\ln(x+1) + c_2'(x) = 0$, we have $c_2'(x) = \ln(x+1)$ and $c_2(x) = \int \ln(x+1)dx = (x+1)(\ln(x+1)-1) + D_2$, where D_2 is an arbitrary real constant. Up so far, we determined $c_1(x)$ and $c_2(x)$, and the general solution to the inhomogeneous ODE is $y_g(x) = c_1(x)\ln(x+1) + c_2(x) = (-x+D_1)\ln(x+1) + (x+1)(\ln(x+1)-1) + D_2 = (D_1-1)\ln(x+1)-(x+1)+D_2$. We can rewrite this solution as $y_g(x) = D_3\ln(x+1)-x+D_4$ by denoting $D_3 = D_1 - 1$ and $D_4 = D_2 - 1$, where D_3 and D_4 are still arbitrary constants.

(Note: If you do not how to obtain $\int \ln z dz = z(lnz - 1) + c$. by the integration by parts method, please check the video `https://www.youtube.com/watch?v=jYLoR9kPB2U`).

As the initial conditions include the derivative $y'(x)$ at $x = 0$, thus we first differentiate our general solution and have $y'(x) = \frac{D_3}{1+x} - 1$. Finally, using the initial conditions, $y(0) = D_3\ln(0+1) - x + D_4 = D_4 = 0$ and $y'(1) = \frac{D_3}{1+1} - 1 = 0$, we have $D_3 = 2$ and $D_4 = 0$, which yields the solution to this BVP as

$$y(x) = 2\ln(x+1) - x$$