Queen Mary University of London

MTH5123 **Differential Equations**

Formative Assessment Week 5 part **G.Bianconi**

- This Formative Assessment consists of three parts:
  - **I.** Practice problems. You will get help on this Formative Assessment in the tutorial of week 3. You should work on this before you go to this session.
  - **II.** Mock Quiz Week 5.
  - **III.** Exploration problems (to help you understand concepts discussed during lecture, not optional and examinable)
- A selection of solutions to the listed problems will be posted on QMPlus by the end of Week 5. You are expected to seek solutions to the remaining problems using the Reading List and making use of the tutorial sessions.
- I encourage all students to learn and check their computational answers using math softwares such as Mathematica, MATLAB, etc. Using numerical software is a fun practice and will help you to visualise your solutions (– sketching solutions will be tested in the final exam).

I. Practice Problems

Question A is for learning content of week 3 - 4. Since many of you think Picard-Lindelöf Theorem is difficult, we add another question here. Since we already explained this in details in week 3 session 2 & 4, week 4 session 2 & 4, we will not discuss this question in our lectures, but you can discuss with your tutors in details again in your week 8 tutorials after the reading week.

A. Consider the initial value problem (IVP) $y' = \frac{1}{2}y^{-1}$ ($y \in \mathbb{R}$), $y(0) = 0$.

1) Use the Picard-Lindelöf Theorem to justify existence and uniqueness of the solution to this ODE (without exhibiting the solution)

2) Now solve the IVP. Find and sketch all possible solutions if the solution is not unique.

3) Change the initial condition to $y(0) = b$ where $b \neq 0$, graph the solution of this new IVP.

B. Assuming $x > 0$ write down the general solution to the Euler-type equations

1) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$

2) $x^2y'' - xy' - 3y = 0$.

C. Find the general solutions of the second order inhomogeneous differential equations:

1) $y'' + 6y' + 8y = -3e^{-x}$

2) $y'' + 7y' + 6y = 10\sin(2x)$

D. Solve the following initial value problem:

$$y'' + 4y' + 5y = 1 - 5x, \quad y(0) = 0,\ y'(0) = -1$$

E. Find the general solution of the second-order inhomogeneous differential equation using variation of paramter

$$y'' + 3y' + 2y = e^{-2x}\cos x\ .$$

Note:

$$\int e^{\alpha x}\cos\beta x\,dx = \frac{\beta}{\alpha^2+\beta^2}e^{\alpha x}\left(\sin\beta x + \frac{\alpha}{\beta}\cos\beta x\right), \quad \alpha \neq \pm i\beta\ ,$$

whereas for $\alpha = \pm i\beta$ it holds

$$\int e^{\pm i\beta x}\cos\beta x\,dx = \frac{x}{2} + \frac{1}{4\beta}\sin 2\beta x \mp \frac{i}{4\beta}\cos 2\beta x\ .$$

II. Homework

Train yourself for Coursework 1 by answering Mock Quiz Week 5.

III. Further Exploration: The application of ODEs on Newton's Second Law

We consider a problem of great importance for applications: the motion of a mass attached to an elastic string under the influence of a periodic driving force, which we have discussed the equations in the lecture (week 1) about Newton's Second Law. Here, we explore the solutions using the methods we learned in week 4 and 5.

Motion under periodic driving force and the resonance phenomenon

We consider differential equations for functions $y(t)$ of an independent variable time $t \in [0, \infty)$. Let us recall that for a point mass m moving along a vertical coordinate y under the influence of a force f *Newton's Second Law* is **mass** $\times$ **acceleration** $=$ **force**. This yields a second-order differential equation

$$m\ddot{y} = f(t, y) \ ,$$

where we will assume for simplicity that there is no friction in the system. Thus, the force f depends on time and position but not on velocity $\dot{y}$. To uniquely determine the motion of this system one has to specify *initial conditions*, which here are the initial value of the coordinate $y(t = 0) = y_0$ and the value of initial speed (velocity) $\dot{y}(t = 0) = v_0$. The

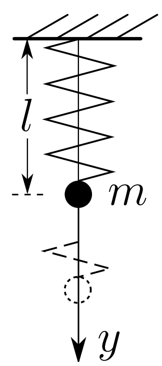

Abbildung 1: Sketch of a spring-mass system.

simplest system of this type is represented by a point mass m attached to the loose end of an elastic spring of length l, with the other end of the spring fixed to a ceiling; see Fig. 1. To keep our considerations as simple as possible we also neglect any additional external driving force acting on the mass. Measuring the coordinate y from the ceiling downwards, the mass is then subject to a force equal to the sum of only two contributions: the position-independent **gravity force** $f_g = mg$ and the position dependent **elastic force** $f_{el} = -k(y - l)$ (Hooke's law of elasticity).

Question: Using the constant parameters k, m, g, l, write down the second ODE linear ODE of the position of the point mass y over time t for the above system. Find the general solution to this ODE, and the solution to the IVP with the initial position and speed at $t = 0$ as $y(0) = y_0, \dot{y}(0) = 0$. Here, y_0 is a given constant number.
(Hint: you first need to identify the variable and independent variable. The ODE can be solved by second order linear ODE methods.)