Essential Foundation Mathematical Skills

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Overview

ILOs

→ Today's lecture is on **Estimation**;

After today's lecture, you are expected to understand the concepts of Writing numbers in base of 10 and Estimating numbers.

Introduction

Definition (Estimation or Rounding or Approximation)

Estimation is when we use approximate values in a calculation to give an approximate, predicted answer rather than an exact answer.

$\rightarrow$ Estimating makes calculations more manageable by using round numbers.

Writing numbers as powers of 10

- Numbers are typically written in base 10, using both positive and negative powers.

  For example:

  $$x = \mathbf{302.0574} = \mathbf{3}\cdot 10^2 + \mathbf{0}\cdot 10^1 + \mathbf{2}\cdot 10^0 + \mathbf{0}\cdot 10^{-1} + \mathbf{5}\cdot 10^{-2} + \mathbf{7}\cdot 10^{-3} + \mathbf{4}\cdot 10^{-4},$$

  that is:
  $302 < x < 303$ (*precise expression*) and $x \approx 302$ (*vague expression*).

- Multiplication of a number by $\mathbf{10^n}$ leads to shifting the number's decimal point $\mathbf{n}$ times:

  To the left direction, for $n < 0$.

  or

  To the right direction, for $n > 0$.

  For example:
  $123.45 \cdot 10^{-4} = 0.012345$

  For example:
  $123.45 \cdot 10^{4} = 1234500.$

Writing numbers as powers of 10

Definition (Scientific notation)

The **scientific notation** of a number $x \neq 0$ is given by:

$$x = \pm a \cdot 10^k, \qquad \text{with } k \in \mathbb{Z} \text{ and } 1 \leq a < 10.$$

For example:
$0.00432 = 4.32 \cdot 10^{-3}$,
$4100000 = 4.1 \cdot 10^6$, etc.

→ The simplest estimates occur with expressions where some terms are much smaller than others, and therefore can be neglected.

For example:

$4.15 \cdot 10^2 + 3 \cdot 10^{-5} = 415.00003 \approx 415$ (i.e. $3 \cdot 10^{-5}$ is neglected).

$4.15 \cdot 10^9 + 3 \cdot 10^2 = 4150000300 \approx 4150000000$ (i.e. $3 \cdot 10^2$ is neglected).

Note that the quantities that we neglect in this example may be vastly different in magnitude, but they give the same **relative contribution**,
i.e. the difference between the powers of 10 is the same: $2 - (-5) = 9 - 2 = 7$.

Writing numbers as powers of 10

Exercises

Write as a decimal number.

0.81×10^0; $0.81/10^0$

1.1×10^2; 1.1×10^{-2}

653×10^{-3}; 0.041×10^7

$37.501 + 10^{-2}$; $20000 + 9 \times 10^2$

$10^5 + 10^{-5}$; $2 \times 10^6 - 10^3$

$6.999 + 10^{-3}$; $3.005 + 50 \times 10^{-4}$

$\frac{123}{100}$; $\frac{123}{10000}$

$\frac{4000}{10^5}$; $\frac{3.3 \times 10^{-3}}{10^{-4}}$

Sort in ascending order.

0.2, 0.03, 0.004, 0.0005

1001001, 1000101, 1010001, 1000011

0.1, 0.09, 0.11, 0.101

67.9, 67.909, 67.8999, 68.01

Estimate of x with a given number n of significant digits of accuracy

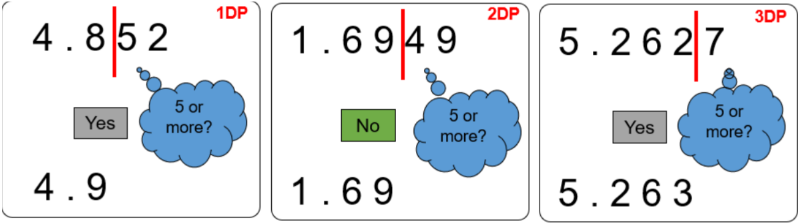
If there are no obvious terms to neglect in a given expression, we have to **specify the accuracy**.

Let $x = 0.02799101\ldots$
By an estimate of x with a given number n of significant digits of accuracy, we mean the following:

n	estimate
1	$0.02 \leq x < 0.03$
2	$0.027 \leq x < 0.028$
3	$0.0279 \leq x < 0.0280$
4	$0.02799 \leq x < 0.02800$
5	$0.027991 \leq x < 0.027992$

$\rightarrow$ When determining significant digits, the leading zeros are **not** counted.
This is evident using scientific notation:
$x = 0.02799101\ldots \approx 2.79 \cdot 10^{-2}$.

Estimate of x with a given number n of significant digits of accuracy



Number	To 1DP	To 2DP	To 3DP
63.4721			
87.6564			
149.9875			
3.54029			
0.59999			

Estimate of x with a given number n of significant digits of accuracy

Number	To 1DP	To 2DP	To 3DP
63.4721	63.5	63.47	63.472
87.6564	87.7	87.66	87.656
149.9875	150.0	149.99	149.988
3.54029	3.5	3.54	3.540
0.59999	0.6	0.60	0.600

Estimate of x with a given number n of significant digits of accuracy

The decimal representation of a number with a **finite** number of digits is **non**-unique.

The simplest example is:

$$1 = 1.0000000000\ldots = 0.9999999999\ldots,$$

where we exploit the fact that a number with finitely many digits really has infinitely many trailing zeros.

More examples:

$30 = 30.000\ldots = 29.999\ldots$
$700100 = 700100.000\ldots = 700099.999\ldots$
$2.003 = 2.003000\ldots = 2.002999\ldots$
$10^{-3} = 0.001000\ldots = 0.000999\ldots$

Subtracting a (relatively) small quantity from a number with finitely many decimals leads to a related phenomenon.

Estimate of x with a given number n of significant digits of accuracy

Exercises

Estimate x to the accuracy indicated in square brackets.

$x = 3 \times (33 \times 10^5 + 1)$ [2]

$x = 2 \times \frac{90}{60001} \times 10^4$ [2]

$x = 4 \times 10^5 + 550 \times 10^3 + 501 \times 10^2$ [2]

$x = \frac{10^1 - 10^{-1}}{10^3}$ [2]

$x = \frac{22 \times 10^{-3} + 20.02 \times 10^{-2}}{2}$ [2]

$x = 10^8 + 15 \times 10^6 + 85 \times 10^4$ [2]

$x = 1000 \times (10^{-1} + 2 \times 10^{-2} + 3 \times 10^{-3} + 4 \times 10^{-4})$ [3]

$x = 10^0 + 10^{-5} - 10^{-10}$ [1]

$x = 10^0 + 10^{-5} - 10^{-10}$ [7]

The coarsest estimates

The **coarsest** estimates are exponential: They provide order of magnitude information.

Let n be a non-negative integer, and consider the following inequalities:

1. $10^{n-1} < x < 10^n$.
   The integer part of x has n decimal digits.
   In other words, 10^{n-1} is the largest power of 10 smaller than x and 10^n is the smallest power of 10 larger than x.

2. $10^{-n-1} < x < 10^{-n}$.
   The fractional part of x has $n+1$ leading zeros, etc.

In the following examples, we have $n = 5$:

$10^4 < 12345.333 < 10^5$
$10^{-6} < 0.00000997 < 10^{-5}$

The coarsest estimates

Exercises

Determine the largest power of 10, smaller than the given number.

5; 12; 99

101777; 9999; 1073741824

101×102; 44×10^2; 8000×10^{-2}

2^4; 2^7; 2^{20}

60×170; 31^2; 32^2

$\frac{201}{2} \times 10$; $\frac{199}{2} \times 10$

$\frac{401}{20} \times \frac{501}{100}$; $\frac{8001}{20} \times \frac{51}{20}$

Determine the largest power of 2, smaller than the given number.

5; 12; 15

31; 32; 33

150; 200; 250

300; 400; 500

1000; 2000; 3000

The coarsest estimates

Exercises

Determine the number of decimal digits in the integer part of the given number.

$$\frac{12345}{2}; \qquad \frac{12345}{10}; \qquad \frac{12345}{1234}$$

$$\frac{19}{5} \times 300; \qquad \left(\frac{201}{2}\right)^2; \qquad \frac{1000^2}{3^2}$$

$$\frac{10^5}{\pi^4} \times \frac{1}{10^{-2}}$$

$$\left(\frac{41}{2}\right)^2 \times \frac{2^{20}}{10^7} \times \frac{5}{2}$$

Examples and exam-style questions

Question: Estimate $x = \frac{70 \times 10^3 - 50.01 \times 10^2}{8 \times 10^4}$

Question: Estimate $x = 5 - 10^{-4} + 10^{-2}$.

Question: Determine how many of the following inequalities are correct:

$$6 > \sqrt{37}, 3 < 2\sqrt{2}, 4 > \sqrt{3}\sqrt{5}, 5 < \sqrt{3}\sqrt{8}$$