Essential Foundation Mathematical Skills

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Overview

ILOs

→ Today's lecture is on **Square roots**;

After today's lecture, you are expected to understand the concepts of Square roots and Arithmetic of Square roots.

Introduction

The most famous square root of all is:

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769\ldots$$

Very little is known about its digits, besides the fact that they will never terminate or repeat!

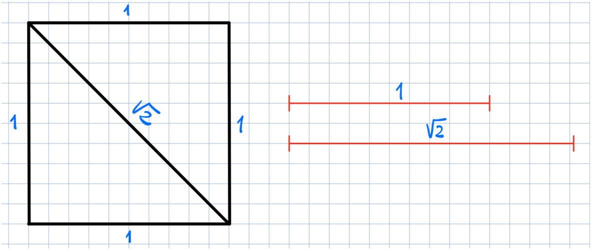

Figure 1: Geometric representation of $\sqrt{2}$.

Square roots

Definition (Square Root)

We define the **square root** of $a \geq 0$ as $\sqrt{a}$, such that $(\sqrt{a})^2 = a$.

Note that:

- $a^{\frac{1}{2}} = \sqrt[2]{a} = \sqrt{a}$.
- $a^{\frac{1}{3}} = \sqrt[3]{a}$.
- $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Basic square roots identities:

1. $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}, \quad a, b \geq 0 \quad \implies \quad \sqrt{a^2} = \sqrt{a} \cdot \sqrt{a} = (\sqrt{a})^2 = a$.
2. $\sqrt{\frac{1}{a}} = \frac{1}{\sqrt{a}}, \quad a \geq 0$.

Combining the 2 identities above, we also get:

$$\sqrt{\frac{a}{b}} = \sqrt{a \cdot \frac{1}{b}} = \sqrt{a} \cdot \sqrt{\frac{1}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, \quad a, b \geq 0.$$

Square roots

Exercises

1. Calculate the following: $\sqrt{\frac{25}{16}}$, $\frac{\sqrt{625}}{11} \times \frac{14}{\sqrt{25}} \times \frac{11}{\sqrt{196}}$.
2. Rationalise the expression: $\frac{1}{\sqrt{5}-\sqrt{3}}$.
3. Simplify the expression: $\sqrt{7 - 2 \cdot \sqrt{6}}$.

Square roots

Definition (Radical or Surd, Radicand)

The expression $\sqrt{a}$ is also called a **radical** or a **(quadratic) surd**.

a is called the **radicand**.

Definition (Square-free Integer)

An integer is **square-free** if it has no square divisors, that is, if all the exponents of the primes in its prime factorization are equal to one.

For example:

$210 = 2 \cdot 3 \cdot 5 \cdot 7$ is a square-free integer, while

$120 = 2^3 \cdot 3 \cdot 5$ is not.

$\rightarrow$ All primes are square-free integers.

Arithmetic of Square roots

- A convenient representation of a radical is obtained by extracting all squares, leaving a square-free kernel under the square root sign.

  For example:

  $$\sqrt{125} = \sqrt{5^3} = 5 \cdot \sqrt{5}, \quad \sqrt{84} = \sqrt{2^3 \cdot 3 \cdot 7} = 2 \cdot \sqrt{21}.$$

- Radicals can be removed from the denominator via **rationalization**:

  $$\sqrt{\tfrac{a}{b}} = \tfrac{\sqrt{a}}{\sqrt{b}} = \tfrac{\sqrt{a}\cdot\sqrt{b}}{\sqrt{b}\cdot\sqrt{b}} = \tfrac{\sqrt{ab}}{\sqrt{b^2}} = \tfrac{\sqrt{ab}}{b},$$

  $$\tfrac{1}{a+\sqrt{b}} = \tfrac{1}{a+\sqrt{b}} \times \tfrac{a-\sqrt{b}}{a-\sqrt{b}} = \tfrac{a-\sqrt{b}}{(a+\sqrt{b})(a-\sqrt{b})} = \tfrac{a-\sqrt{b}}{a^2-b}.$$

  For example:

  $$\tfrac{1}{\sqrt{120}} = \tfrac{\sqrt{120}}{\sqrt{120}\cdot\sqrt{120}} = \tfrac{\sqrt{120}}{120} = \tfrac{\sqrt{30}}{60},$$

  $$\sqrt{\tfrac{45}{28}} = \tfrac{3}{2}\sqrt{\tfrac{5}{7}} = \tfrac{3}{2}\tfrac{\sqrt{5}}{\sqrt{7}} = \tfrac{3}{2}\tfrac{\sqrt{5}\cdot\sqrt{7}}{\sqrt{7}\cdot\sqrt{7}} = \tfrac{3}{2}\tfrac{\sqrt{5}\cdot\sqrt{7}}{7} = \tfrac{3\cdot\sqrt{5\cdot 7}}{2\cdot 7} = \tfrac{3\cdot\sqrt{35}}{14}.$$

Arithmetic of Square roots

- To estimate the size of an expression involving square roots, we sandwich it between two consecutive integers and then square it.

  For example:

  $$\begin{aligned} n < 3\cdot\sqrt{13} < n+1 \quad &\Longrightarrow\ n^2 < 9\cdot 13 < (n+1)^2 \\ &\Longrightarrow\ n^2 < 117 < (n+1)^2 \\ &\Longrightarrow\ n = 10 \qquad\qquad (\text{because } 10^2 < 117 < 11^2). \end{aligned}$$

  Or, for example, if we want to estimate $\frac{29-3\cdot\sqrt{13}}{7}$, we use the previous example and we have:

  $$\begin{aligned} 10 < 3\cdot\sqrt{13} < 11 \quad &\Longrightarrow\ -11 < -3\cdot\sqrt{113} < -10 \\ &\Longrightarrow\ 29-11 < 29-3\cdot\sqrt{13} < 29-10 \\ &\Longrightarrow\ \frac{29-11}{7} < \frac{29-3\cdot\sqrt{13}}{7} < \frac{29-10}{7} \\ &\Longrightarrow\ 2+\frac{4}{7} < \frac{29-3\cdot\sqrt{13}}{7} < 2+\frac{5}{7}. \end{aligned}$$

Examples and exam-style questions

Exercise: Simplify the following expressions to the form $r \cdot \sqrt{d}$, where r is a fraction and d is a square-free integer:

- $\sqrt{11} - \frac{1}{\sqrt{11}}$,
- $5 \cdot \sqrt{3} + \frac{7}{\sqrt{3}}$,
- $\sqrt{6} \cdot \sqrt{21} - \sqrt{\frac{32}{7}}$,
- $\sqrt{\frac{7}{3}} - \sqrt{\frac{3}{7}} + \sqrt{\frac{36}{21}}$.

Examples and exam-style questions

Exercise: Simplify the following expressions to the form $s + r \cdot \sqrt{d}$, where s, r are fractions and d is a square-free integer:

- $\left(\sqrt{60} - \frac{\sqrt{15}}{2}\right)^2$,
- $\sqrt{3}(\sqrt{3} - 1)^2$,
- $(\sqrt{5} + \sqrt{6})^3$,
- $\frac{2+\sqrt{8}}{(\sqrt{2})^3}$,
- $\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{3-\sqrt{5}}{2}\right)$,
- $\left(1 - \frac{1}{\sqrt{5}}\right) \cdot (\sqrt{5} - 1)^2$,
- $\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2$,
- $\left(\sqrt{60} - \frac{\sqrt{15}}{2}\right)^2$,
- $(1 - \sqrt{2})(1 + \sqrt{2})$.

$$\frac{-\sqrt{3}+1}{\sqrt{3}-3} - \frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{2}+\sqrt{3}} \sqrt{\frac{1}{2}}$$

$$\frac{1}{\sqrt{80}} - \frac{\sqrt{5}}{3\sqrt{5}-7}$$

$$\frac{4-\sqrt{2}\sqrt{3}}{-\sqrt{9}+\sqrt{6}}$$

Examples and exam-style questions

Question: Find how many of the following inequalities are correct:

(i) $7 > \sqrt{48}$ (ii) $2\sqrt{3} < \sqrt{13}$ (iii) $11 > \sqrt{120}$ (iv) $6 < \sqrt{35}$

Question: Simplify, eliminating radicals in the denominator

$$\frac{\sqrt{3}}{\sqrt{5}-2}$$

Question: Simplify, eliminating radicals at denominator,

$$\sqrt{\frac{7}{14}+\frac{1}{5}-\frac{5}{4}+1}$$