Essential Foundation Mathematical Skills

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Overview

ILOs

$\rightarrow$ Today's lecture is on **Fractions**;

After today's lecture, you are expected to understand the concepts of Fractions and Arithmetic of Fractions.

Introduction

History: The earliest fractions were invented by the Egyptians and they were reciprocals of integers representing one part of two, one part of three, one part of four and so on.

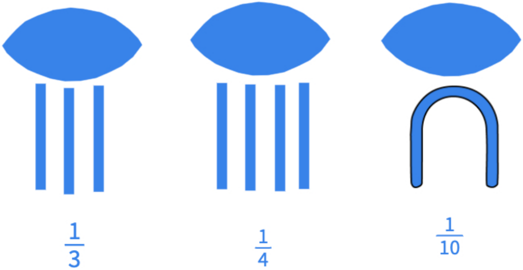

Figure 1: Egyptian symbols representing fractions.

Fractions

Definition (Fraction)

A **fraction** is the ratio of two integers, the numerator and the denominator, where the denominator is non-zero.

One can write $\frac{n}{d}$, with $n \in \mathbb{Z}$ and $d \in \mathbb{Z} - \{0\}$.

Examples: $\frac{1}{4}, \frac{3}{6}, \frac{-4}{11}, \frac{225}{-30}, \frac{0}{328746}, -\frac{7}{10^4}$, etc.

→ An integer may be thought of as a fraction with denominator 1:

$$5 = \frac{5}{1}, \quad -40 = \frac{-40}{1} = -\frac{40}{1}, \quad 0 = \frac{0}{1}, \quad \text{etc.}$$

→ Properties of fractions:

- $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$.
- $\frac{a}{b} = \frac{c}{d} \iff a \cdot d = b \cdot c$.
- $\frac{a}{b} < \frac{c}{d} \iff a \cdot d < b \cdot c$.

→ Note that $a\% = \frac{a}{100}$.

For example:

$$20\% = \frac{20}{100} = \frac{1}{5}.$$

Fractions

Definition (Reduced Fraction)

A fraction is **reduced** if the numerator and denominator are relatively prime and the denominator is positive.

In other words, a fraction $\frac{n}{d}$, with $n \in \mathbb{Z}$ and $d \in \mathbb{Z} - \{0\}$ is **reduced** iff:

1. $\gcd(n, d) = 1$,
2. $d > 0$.

Examples:

- $\frac{19}{54}$ is reduced.
- $\frac{36}{54}$ is **not** reduced, because $\frac{36}{54} = \frac{2 \cdot 18}{3 \cdot 18} = \frac{2}{3}$.

  $\rightarrow$ Note that $\gcd(36, 54) = 18 \neq 1$.

  $\rightarrow$ $\frac{2}{3}$ is the **reduced form** of $\frac{36}{54}$.

To reduce a fraction, one divides numerator and denominator by their greatest common divisor (GCD).

Fractions

- Every fraction can be written as the sum of an integer and a fraction lying in $[0, 1)$, which are its **integer part** and **fractional part**, respectively.

  For example:

  $$\frac{19}{5} = \frac{15+4}{5} = \frac{15}{5} + \frac{4}{5} = 3 + \frac{4}{5}, \quad -\frac{35}{6} = -5 - \frac{5}{6}, \quad \frac{6}{21} = 0 + \frac{6}{21} = 0 + \frac{2}{7}, \quad \text{etc.}$$

- The **fractional part** of an integer number is zero.

  For example:

  $$7 = 7 + \frac{0}{7}, \quad -2 = -2 - \frac{0}{2}, \quad 2023 = 2023 + \frac{0}{2023}, \quad \text{etc.}$$

- Given a fraction $\frac{n}{d}$, with $n \in \mathbb{Z}$ and $d \in \mathbb{Z} - \{0\}$, we have that:

  $$n \div d = q \cdot n + r. \tag{1}$$

  The **integer** and **fractional parts** of the fraction $\frac{n}{d}$ are the quotient (q) and the remainder (r), respectively, of division 1.

Fractions

- Knowledge of the **integer part** of a fraction allows us to make a rough estimation of its size. For example:

  $$\frac{7}{3} = 2 + \frac{1}{3} \implies 2 < \frac{7}{3} < 3.$$

  Knowledge of the **fractional part** of a fraction allows us to determine which integer it is closer to. For example:

  $\frac{1}{3} < \frac{1}{2}$, so $\frac{7}{3}$ is closer to number 2.

- A number with terminating decimals can be written as a reduced fraction. For example:

  $$7.1 = \frac{71}{10}, \quad 7.04 = \frac{704}{100} = \frac{176}{25}, \quad 5.234532 = \frac{5234532}{1000000} = \frac{1308633}{250000}, \quad \text{etc.}$$

  However, most fractions do not have terminating decimals; a reduced fraction has terminating decimals only when 2 and 5 are the only primes which divide the denominator. For example:

  $$\frac{1}{1280} = \frac{1}{2^8 \cdot 5} = 0.00078125, \quad \text{but} \quad \frac{1}{7} = 0.142857\ 142857\ 142857\ 142857\ \ldots$$

Arithmetic of fractions

Addition: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

Subtraction: $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

Multiplication: $\frac{a}{b}\times\frac{c}{d}=\frac{ad}{bd}$.

Division: $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

Exponentiation: $\left(\frac{a}{b}\right)^{e} = \frac{a^e}{b^e}$ and $\left(\frac{a}{b}\right)^{-e} = \frac{b^e}{a^e}$.

$\rightarrow$ Recall the following properties:

- $x^{-n} = \frac{1}{x^n}$.
- $\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}$ and $\left(\frac{a}{b}\right)^{-n} = \frac{1}{(\frac{1}{b})^n} = \left(\frac{b}{a}\right)^{n}$.

Arithmetic of fractions

- When adding/subtracting fractions whose denominators are **not** relatively prime (their gcd is $\neq 1$), one should use the reduced formula:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a\frac{L}{b} \pm c\frac{L}{d}}{L},$$

where L=lcm(b,d).

For example, we have that:

$$\frac{8}{21} - \frac{9}{28} = \frac{8 \cdot 28 - 9 \cdot 21}{588} = \frac{35}{588}, \quad \text{where } 588 = 21 \cdot 28,$$

but with the use of the **reduced** formula, we have:

$$\frac{8}{21} - \frac{9}{28} = \frac{8 \cdot 4 - 9 \cdot 3}{84} = \frac{5}{84}, \quad \text{where } \mathrm{lcm}(21, 28) = 84.$$

Arithmetic of fractions

- Before multiplying fractions, check the possibility of cross-simplification:

$$\frac{\cancel{a}}{b} \times \frac{c}{\cancel{a}} = \frac{c}{b}.$$

For example,

$$\frac{\cancelto{1}{40}}{51} \times \frac{17}{\cancelto{2}{80}} = \frac{17}{51 \cdot 2} = \frac{1}{6}.$$

- Make sure you are fully comfortable with the behaviour of fractions with respect to sign change, e.g.:

$$-\frac{3-2}{11-7} = -\left(\frac{3-2}{11-7}\right) = \frac{-(3-2)}{11-7} = \frac{3-2}{-(11-7)} = \frac{2-3}{11-7} = \frac{3-2}{7-11} = \dots$$

Arithmetic of fractions

- When evaluating arithmetical expressions involving several operators, multiplication and division are performed **before** addition and subtraction. For example:

$$\frac{a}{b} - \frac{c}{d} \div \frac{e}{f} = \frac{a}{b} - \left(\frac{c}{d} \div \frac{e}{f}\right) = \frac{a}{b} - \frac{cf}{de} = \dots$$

- Expressions within parentheses are evaluated first. With nested parentheses, evaluation begins from the innermost one. For example:

$$\left[\frac{1}{4} \times \left(2 - \frac{2}{3}\right)\right] \div 5 - \left(\frac{2}{5} - \frac{7}{3}\right) = \left[\frac{1}{4} \times \frac{4}{3}\right] \times \frac{1}{5} - \left(-\frac{29}{15}\right) = \frac{1}{15} + \frac{29}{15} = 2.$$

Keep fractions simplified at every stage of a calculation!!!

Examples and exam-style questions

Question: Multiply, cross-simplifying first:

$$\frac{5}{18}\times\frac{6}{11}, \quad \frac{24}{5}\times\frac{81}{30}\times\frac{25}{36}\times\frac{2}{45}.$$

Question: Add, subtract: $\frac{3}{4}+\frac{5}{12}-\frac{1}{20}-\frac{1}{4}$.

Question: Evaluate, eliminating parentheses first:

$$2\times\left(\frac{1}{2}+\frac{1}{4}\right), \quad \left(\frac{3}{10}+\frac{6}{5}\right)\times\left(4-\frac{7}{28}\right), \quad \frac{1}{-1}, \quad \frac{(-1)^2}{-1^2}, \quad -\frac{-1^3}{(-1)^3}, \quad \left(\frac{2}{3}\right)^2,$$

$$\left(\frac{5}{7}\right)^0, \quad \left(\frac{1}{3}\right)^{-2}, \quad \left(2-\frac{1}{3}\right)^3, \quad \left(\frac{5}{2}-\frac{1}{3}\right)^2, \quad \left(-\frac{7}{3}+3\right)^4, \quad \left(-1-\frac{5}{6}\right)^{-2},$$

$$\left(\frac{2}{3}-\frac{1}{6}\right)^{-5}, \quad \left(\frac{6}{8}-\frac{6}{10}\right)^{-3}, \quad \left(\frac{2}{3}\right)^{10}\div\left(\frac{2}{3}\right)^8, \quad \frac{3^5}{2^5}\div\frac{3^6}{2^6},$$

$$\left[\left(\frac{2}{5}\right)^4\times\left(\frac{2}{5}\right)^3\right]^2\div\left(\frac{2}{5}\right)^{12}.$$

Examples and exam-style questions

Question: Evaluate

$$\left(\frac{3}{5}\right)^2 - \left(\left(\frac{1}{5} - \frac{1}{7}\right) \times 7\right)^2$$

Question: Compute the remainder of the following division: $2022 \div 7$

Question: Find how many of the following equalities are correct:

$$44892 \div 10^4 = 4.4892, 5.7689 \times 10^{-4} = 57689$$

$$0.00532 \times 10^4 = 5.32, 3149 \div 10^4 = 3.149$$