Essential Foundation Mathematical Skills

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September 28, 2023

Overview

ILOs

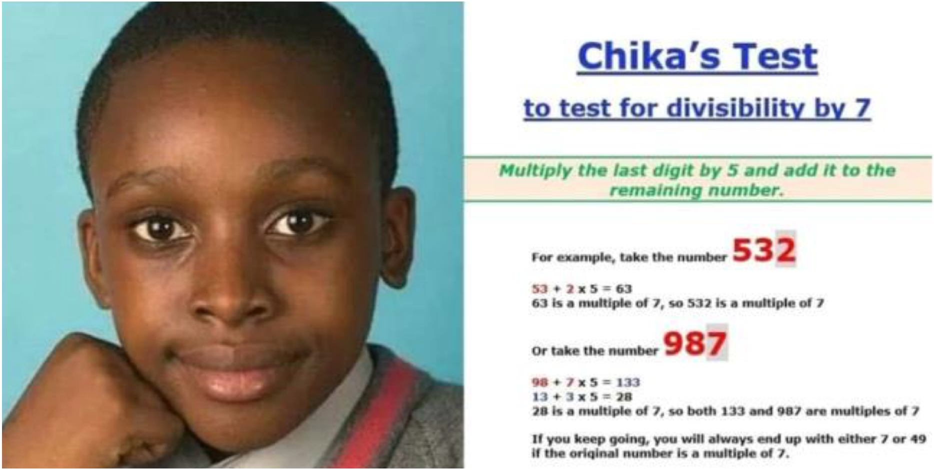
→ Today's lecture is on **Divisibility and Primes**;

After today's lecture, you are expected to understand the concepts of prime decomposition and divisibility of numbers.

Introduction

- The Greek mathematicians were the first to study prime numbers and their characteristics in depth.
- The mathematicians of Pythagoras' school were intrigued by numbers, because of their mystical and numerological characteristics (500 BC to 300 BC).
- They were aware of the concept of primality and had a passion for harmonious and flawless numbers.

Introduction



Divisibility

Definition (Multiplication)

Let a, b, c, d, $\ldots$ be integers ($\in \mathbb{Z}$, both positive/negative).
Notation for multiplication:
$a \cdot b = ab = a \times b$
is the **product** of a and b.

Basic properties:

$a \cdot b = b \cdot a$

$(ab)^c = a^c \cdot b^c$

$a^b \cdot a^c = a^{b+c}$

$(a^b)^c = a^{bc}$

Divisibility

Definition (Divisibility)

Let a and d be integers.

We say that d **divides** a if we can write $a = q \cdot d$ for some integer q.

The integer $q = a \div d = a{:}d = \frac{a}{d}$ is called the **quotient** of the division of a by d.

There may be the case when we have remainder:
If $d \neq 0$, we can always find integers q and r, such that $a = q \cdot d + r$, with $0 \leq r <| d |$.

The symbol $| \cdot |$ denotes the absolute value.

The integer r is called the **remainder** of the division of a by d.

$\rightarrow$Don't forget: r is always non-negative.

Primes

Definition (Prime)

A **prime** is an integer greater than 1 which is divisible only by itself and 1.

The sequence of primes:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots$$

is infinite.

To test whether or not a positive integer n is prime, we check divisibility by all primes p with $p \neq \sqrt(n)$;

if n is not divisible by any of them, then n is prime.

Primes

Definition (Prime Decomposition factorization)

Every integer greater than 1 can be decomposed as a product of primes, and this process is called **prime decomposition factorization**.

The above decomposition is unique, apart from rearrangement of factors.
For example:

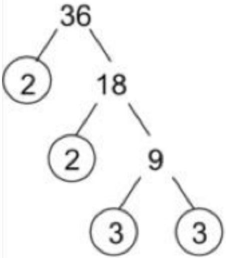
$$120 = 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 5 \cdot 3 \cdot 2^3 = 2^3 \cdot 3 \cdot 5 = \dots$$

Definition (Composite)

A **composite** is an integer that can be decomposed as a product of more than two primes.

All composite numbers can be expressed as a product of prime numbers.

Examples and exam-style questions



1. Split the number into 2 factors that multiply to make the starting number.
2. Circle any factors that are prime numbers.
3. Continue to split any numbers that are not prime.

10	12	14	15	20	30
28	44	48	56	60	72
112	140	240	310	144	192

Examples and exam-style questions

Answers

$10 = 2 \times 5$	$12 = 2^2 \times 3$
$14 = 2 \times 7$	$15 = 3 \times 5$
$20 = 2^2 \times 5$	$30 = 2 \times 3 \times 5$

$28 = 2^2 \times 7$	$42 = 2 \times 3 \times 7$
$48 = 2^4 \times 3$	$56 = 2^3 \times 7$
$60 = 2^2 \times 3 \times 5$	$72 = 2^4 \times 3^2$

$112 = 2^4 \times 7$	$140 = 2^2 \times 5 \times 7$
$240 = 2^4 \times 3 \times 5$	$310 = 2 \times 5 \times 31$
$144 = 2^4 \times 3^2$	$192 = 2^6 \times 3$

Examples and exam-style questions

Question: Compute the remainder of the following division: $1734 \div 7$

Question: Write 792 as a product of prime numbers.