Probability

"It will probably rain for the graduation"

"Our team has a better chance of winning the game tomorrow"

"Stuart is more likely than George to be a student rep this year".

What do all these expressions convey?

These sentences express an opinion that one outcome is more likely than another

but

in none of them is there any attempt to state by how much.

However, it might be interesting, useful, necessary to adopt a quantitative approach

or

to find a way of calculating the probability or the likelihood of the event of interest.

e.g. the probability or likelihood of rain.

Probability

Theoretical | Empirical

Theoretical Probability

A lot of ideas of probability are based on symmetry.

e.g. If you toss a coin repeatedly, how often will you get heads?

e.g. If you roll a dice, how often will you get a four?

e.g. If you roll two dice several times, how often will you get two sixes?

Event

is the total number of outcomes (a set of outcomes) that has a probability assigned to it.

Outcome

is the result of a random experiment.

Note: The set of all possible outcomes is called the sample space.

Outcome:

A result of a random experiment.

Sample Space:

The set of all possible outcomes.

Event:

A subset of the sample space.

Theoretical Probability

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

Experimental Probability

repeat the experiment and observe the outcomes

$$P(\text{event}) = \frac{\text{number of times event occurs}}{\text{total number of trials}}$$

e.g. "A coin is tossed 15 times. Head is recorded 10 times and tail is recorded 5 times."

$P(\text{Head}) =$

$P(\text{Tail}) =$

What is the theoretical probability of the head when "A coin is tossed"?

Be careful to ensure that you have a complete list of all possible outcomes of the event under consideration.

Example:

What is the probability of getting five sixes when five dice are rolled?

Example:

What is the probability that there will be at least one head in five tosses of a fair coin?

Axioms of Probability

- For any event A, $P(A) \geq 0$.
- Probability of the sample space Ω is $P(\Omega) = 1$.
- If $A_1, A_2, A_3, \ldots, A_n$ are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n)$$

Example:

On my way to campus every day I often see squirrels.

In four weeks my observations were the following:

Number of squirrels seen	Number of occasions
0	0
1	3
2	5
3	7
4	2
5	1
6	0
7	1
8	1

What is the probability that I will see at least two squirrels in my next journey?

Random Experiments - Probabilities - Sets

Consider a sample space Ω and three events A, B and C.

Find the set expression for the following:

1) Among A, B and C, only A occurs.
2) At least one of the events A, B or C occurs.
3) A or C occurs, but not B.
4) At most two of the events A, B or C occur.

Using the axioms of probability, we can prove the following:

1) For any event A, $P(A^c) = 1 - P(A)$.

2) The probability of the empty set ϕ is zero, i.e. $P(\phi) = 0$.

3) For any event A, $P(A) \leq 1$.

4) $P(A \setminus B) = P(A) - P(A \cap B)$

5) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Inclusion - Exclusion ~~formula~~ principle)

6) If $A \subseteq B$, then $P(A) \leq P(B)$.

Example:

We know that:

There is a 60 percent chance that it will rain today.

There is a 50 percent chance that it will rain tomorrow.

There is a 30 percent chance that it does not rain either day.

Find the probabilities below:

1) The probability that it will rain today or tomorrow.

2) The probability that it will rain today and tomorrow.

3) The probability that it will rain today but not tomorrow.

4) The probability that it will rain either today or tomorrow, but not on both days.

Conditional Probability

The main question is:

As you have more information, how should we update probabilities of events?

For instance, if we pick a random day, the probability that it is sunny is 23 percent, i.e.

$P(S) = 0.23$, where S is the event that it is sunny on the randomly chosen day.

Let us choose a random day, but it is cloudy on this chosen day.

We now have this extra piece of information; how do we update the chance that it is sunny on that day?

In other words,

what is the probability that it is sunny given that it is cloudy?

This is the conditional probability of S given that C has occurred, with C denoting the event that it is cloudy on a randomly chosen day.

If A and B are two events in a sample space Ω, then the conditional probability of A given B is given by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ when } P(B) > 0.$$

Therefore ☺,

for three events A, B and C, with $P(C)>0$, we have:

1) $P(A^c|C) = 1 - P(A|C)$

2) $P(\emptyset|C) = 0$

3) $P(A|C) \leq 1$

4) $P(A-B|C) = P(A|C) - P(A \cap B|C)$

5) $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$

6) If $A \subseteq B$, then $P(A|C) \leq P(B|C)$.

Example:

I roll a fair dice. What is the probability of getting an odd number?
What is the probability of getting an odd number given that the outcome is less than or equal to 3?

Independence

Two events A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

Example:

We pick a random number from $\{1, 2, .., 10\}$ and this random number is denoted by η.

Assume that all outcomes are equally likely.

Let A be the event that η is less than 7.

Let B be the event that η is an even number.

Are A and B independent?

The definition of independence can be extended to more than two events, i.e.

let us assume three events A, B and C are independent.
Therefore, **ALL** of the following hold:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap C) = P(A) \cdot P(C)$$

$$P(B \cap C) = P(B) \cdot P(C)$$

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

Please note :

Concept	Meaning	Mathematical Formulae
Disjoint	A and B cannot occur at the same time	• $A \cap B = \emptyset$, • $P(A \cup B) = P(A) + P(B)$
Independent	A does not give any information about B.	• $P(A\|B) = P(A)$ • $P(B\|A) = P(B)$ • $P(A \cap B) = P(A) \cdot P(B)$

Please also note:

When two events A and B are disjoint, if one of them occurs, the other one cannot occur, i.e.

$$A \cap B = \emptyset.$$

Therefore, the event A gives "a lot of" information about the event B and this is the reason that they cannot be independent.

Law of Total Probability

If $B_1, B_2, \ldots$ is a partition* of the sample space Ω, then for any event A,

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \cdot P(B_i)$$

*partition: If $B_1, B_2, \ldots$ is a partition of the sample space Ω, we can write

$$\Omega = \bigcup_i B_i$$, with B_i being disjoint

Therefore,

$$\begin{aligned} A &= A \cap \Omega \\ &= A \cap \left(\bigcup_i B_i\right) \\ &= \bigcup_i (A \cap B_i) \quad \text{(distributive law)} \end{aligned}$$

The sets $A \cap B_i$ are disjoint, since B_i's are disjoint.
Based on the third axiom of probability,

$$P(A) = P\left(\bigcup_i (A \cap B_i)\right) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i) \cdot P(B_i)$$