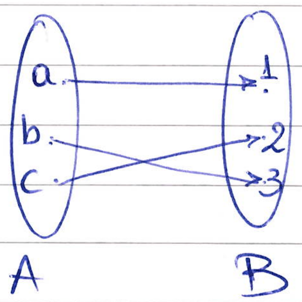
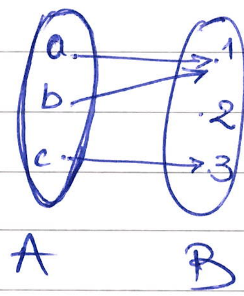
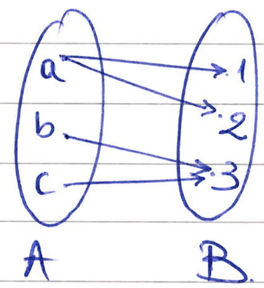
Functions

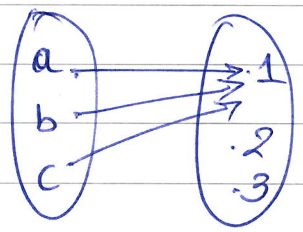
Suppose A and B are two non-empty sets. Then, a function "f" from A to B is a correspondence from A to B such that for EACH $x \in A$, there is a UNIQUE $y \in B$. The set A is called the Domain of the function and the set B is called the target set or the codomain.

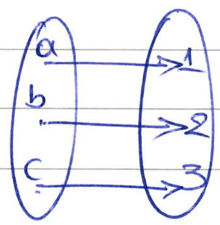
Examples:

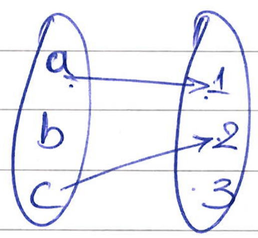












A function can be expressed by means of a mathematical formula.

Suppose a function which transforms each real number into its square.

We write it in different forms:

- $f(x) = x^2$, x is the variable and f denotes the function.

- $x \mapsto x^2$, here the barred arrow is read as "goes into"

- $y = x^2$, x is called the independent variable, y is called the dependent variable.

Note: Whenever a function is given by a formula in terms of a variable x, usually the domain of the function is $\mathbb{R}$ and the codomain is also $\mathbb{R}$.

Functions as Relations

A function $f: A \mapsto B$ is a relation from A to B (i.e. a subset of $A \times B$) such that $a \in A$ belongs to a unique ordered pair (a,b).

Real Polynomial Functions

A function $f: \mathbb{R} \mapsto \mathbb{R}$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_i \in \mathbb{R}$ is called a real polynomial function.

The graph of such functions can be made (the function can be plotted) by plotting some of its points and then drawing a smooth curve through these points.

Some special types of functions

Injective functions (One-to-one)

A function $f: S \longmapsto T$ is injective if and only if $\forall\, x_1, x_2 \in S,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

The function $f: S \longmapsto T$ is injective (one-to-one) if different elements in the domain S have distinct images.

Examples:

Surjective Functions (Onto Functions)

A function $f: S \mapsto T$ is onto (surjective) if and only if $\forall y \in T, \exists x \in S, f(x) = y$.

A function $f: S \mapsto T$ is surjective if each element of T is the image of some element of S.

Alternatively, f is surjective if and only if $T = \text{image}(f)$, i.e. the range and the codomain of f are the same.

Bijective Functions

A function is bijective (or a bijection) if it is both injective and surjective.

Examples.

Inverse functions

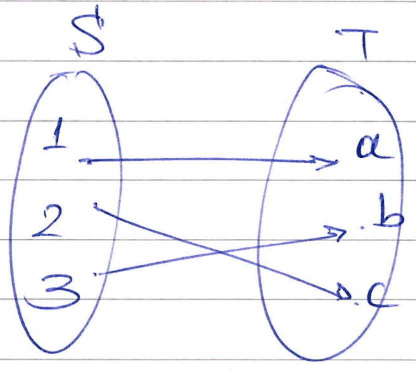
The most important aspect of bijective functions is that they are invertible. In addition, the inverse of a bijection is unique.

Let $f: S \mapsto T$ be a bijection. Then, the inverse function $f^{-1}: T \mapsto S$ is the function given by the following:

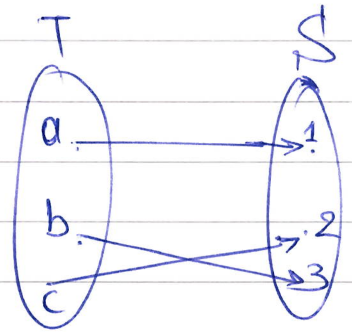
For $y \in T$, $f^{-1}(y)$ is the unique $x \in S$ such that $f(x) = y$. Thus,

$$y = f(x) \iff x = f^{-1}(y).$$

e.g.



$$f = \{(1,a), (2,c), (3,b)\}$$



$$f^{-1} = \{(a,1),(b,3),(c,2)\}.$$

Composition of functions

Consider functions $f: A \mapsto B$ and $g: B \mapsto C$, where the codomain of f is the domain of g.

Then, we define a new function from A to C, called the composition of f and g and written as $g \circ f$ as follows:

$$(g \circ f)(a) = g(f(a))$$

"the function g of f"

"the function g composed with f".

Thus, we find the image of a under f and then the image of $f(a)$ under g.

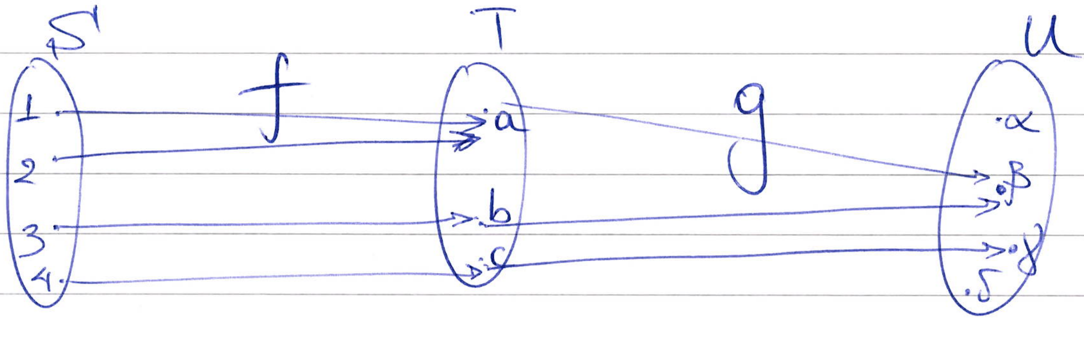
Examples:

Suppose $S = \{1, 2, 3, 4\}$, $T = \{a, b, c\}$, $U = \{\alpha, \beta, \gamma, \delta\}$. Let

$$f = \{(1,a), (2,a), (3,b), (4,c)\}$$

$$g = \{(a,\beta), (b,\beta), (c,\gamma)\}.$$

Then, we have:



So, $g \circ f = \{(1,\beta), (2,\beta), (3,\beta), (4,\gamma)\}$

Alternatively,

$(g \circ f)(1) = g(f(1)) = g(a) = \beta$.

$(g \circ f)(2) = g(f(2)) = g(a) = \beta$

$(g \circ f)(3) = g(f(3)) = g(b) = \beta$.

$(g \circ f)(4) = g(f(4)) = g(c) = \gamma$.

Note:

In general, compositions of functions are not commutative.

Associativity

Let f, g, h be functions. Then,

(i) $(h \circ g) \circ f$ exists $\Leftrightarrow$ $h \circ (g \circ f)$ exists.

(ii) when $(h \circ g) \circ f$ and $h \circ (g \circ f)$ exist, they are equal.