Induction (strong induction)

second form

Let $P(n)$ be a proposition defined on the integers $n \geq 1$ such that:

(i) $P(1)$ is true.

and

(ii) $P(k)$ is true whenever $P(j)$ is true for all $1 \leq j \leq k$.

Then, $P(n)$ is true for every integer $n \geq 1$.

Step 1: Demonstrate the base case

Step 2: Prove the inductive step.

We assume that all of $P(k_0), P(k_0+1), \ldots, P(k)$ are true. Then, we show $P(k+1)$ is true.

Fibonacci Sequence.

$$F_{n+2} = F_{n+1} + F_n \quad , \text{for } n \geq 0, n \in \mathbb{Z}.$$

with $F_1 = F_2 = 1$.

Show that:

$$F_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

Examples (Practice):

- Prove that $n! > 2^n - 1$, for all integers with $n \geq 4$
- Prove that $n^2 > n+3$, for all integers with $n \geq 3$.

Finite Sets and Cardinality

Some important properties:

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \setminus B| = |A| - |A \cap B|$
- $|A \cap B| = |A| + |B| - |A \cup B|$

e.g. $A = \{a, b, c\}$, $B = \{b, d, e, x, z\}$

Then,

$$A \cup B = \{a, b, c, d, e, x, z\}$$

$$A \cap B = \{b\}$$

$$A \setminus B = \{a, c\}$$

$|A| = 3$, $|B| = 5$, $|A \cup B| = 7$, $|A \cap B| = 1$, $|A \setminus B| = 2$

Based on the above-mentioned propertie

$$|A \cup B| = 7 = |A| + |B| - |A \cap B| = 3 + 5 - 1$$

$$|A \setminus B| = 2 = |A| \setminus |A \cap B| = 3 - 1$$

$$|A \cap B| = 1 = |A| + |B| - |A \cup B| = 3 + 5 - 7$$

$\mathcal{P}_5$) If A and B are disjoint sets, i.e. $A \cap B = \emptyset$ then

$|A \cap B| = 0$ and $|A \cup B| = |A| + |B|$

$\mathcal{P}_6$) $|A^c| = |U| - |A|$, where U is the universal set

Cartesian Product of Sets

An ordered pair of elements a and b, where a is the first element and b is the second element, is denoted by (a,b).

In particular,

$$(a,b) = (c,d) \Longleftrightarrow a=c \wedge b=d.$$

The set of all ordered pairs (a,b), where $a \in A$ and $b \in B$ is called the cartesian product of A and B. A and B are two arbitrary sets and the cartesian product of A and B is denoted by $A \times B$, i.e.

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Example:

$A = \{1,2\}$, $B = \{a,b,c\}$

$A \times B =$

$B \times A =$

$A \times A =$

Relations

Let A and B be sets. A binary relation or simply a relation from A to B is a subset of $A \times B$.

Let us suppose R is a relation from A to B.
Then, R is a set of ordered pairs where each first element comes from A and each second element comes from B, i.e

for each pair (a,b) with $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a,b) \in R$; we say "a is related to b" and we write it as aRb.

or

(ii) $(a,b) \notin R$; we say "a is not related to b" and we write it as $a \not R b$.

Note that if R is a relation from a set A to itself, i.e. if R is a subset of $A^2 = A \times A$ then we say that R is a relation on A.

The domain of a relation R is the set of all the first elements of the ordered pairs that belong to R.

The range of a relation R is the set of all the second elements of the ordered pairs that belong to R.

Some important relations:

The empty relation (or void relation): if no element of set A is related (mapped) to any of the elements of set B. It is denoted by $R = \emptyset$.

The universal relation.

If every element of A is related to the set A, i.e. $R = A \times A$.

The identity relation

is denoted by $I = \{(x, x) : x \in A\}$.

Example:

$A = \{1, 2, 3\}$, $B = \{x, y, z\}$ and

R is defined as $R = \{(1,y), (1,z), (3,y)\}$

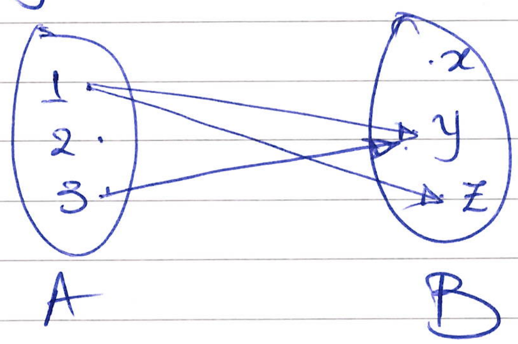
- Then, R is a relation from A to B, as R is a subset of $A \times B$.
- For this relation,

  $1Ry$, $1Rz$, $3Ry$

  but

  $1 \not R x$, $2 \not R y$, $3 \not R z$ etc.
- The domain of R is $\{1, 3\}$ and the range of R is $\{y, z\}$.

A (better ?) way to visualise and understand the relations is to see it pictorially:



$R = \{(1,y),(1,z),(3,$

Inverse Relation

Let R be any relation from a set A to a set B. The inverse of R, denoted by R^{-1} is a relation from B to A consisting of those ordered pairs which, when reversed, belong to R, i.e.

$$R^{-1} = \{ (b,a) \mid (a,b) \in R \}.$$

In the above example, the inverse of $R = \{(1,y), (1,z), (3,y)\}$ is

Types of Relations:

a) Reflective Relations

A relation R on a set A is reflective if aRa for every $a \in A$, i.e. if $(a,a) \in R$ for every $a \in A$.

R is <u>not reflective</u>, if $\exists a \in A$ such that $(a,a) \notin R$.

b) Symmetric and Antisymmetric Relatio

- A relation R on a set A is symmetric if whenever aRb then bRa, i.e. if whenever $(a,b) \in R$ then $(b,a) \in R$.

Thus,

R is not symmetric, if $\exists\ a,b \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$.

- A relation R on a set A is antisymmetri if whenever aRb and bRa, then $a=b$, i.e. if $a \neq b$ and aRb, then $b \not R a$.

c) Transitive Relations :

A relation R on a set A is transitive, if whenever aRb and bRc, then aRc, i.e. whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

Equivalence Relations :

Consider a non-empty set S. A relation R on S is an equivalence relation, if R is reflective, symmetric and transitive.

Totality

$\forall a, b \in S$, aRb or bRa.

Order Relations

(i) $R \subseteq S \times S$ is a partial order, if R is reflexive, antisymmetric and transitive.

(ii) If, in addition, R is total, then we say that R is a total order.

Note that any total order is also a partial order.