Mathematical Induction

Let P be a proposition defined on positive integers ($\mathbb{Z}_+$ or $\mathbb{N}$, depending whether 0 (zero) belongs to the set of natural numbers), i.e. $P(n)$ is either true or false for each $n \in \mathbb{N}$.

Suppose $P(n)$ has the following properties:

(i) $P(1)$ is true.

(ii) $P(k+1)$ is true whenever $P(k)$ is true.

Then, $P(n)$ is true for every positive integer $n \in \mathbb{N}$.

Example:

Let P be a proposition that the sum of the first n odd numbers is n^2, i.e.

$$P(n): \quad 1+3+5+\dots+(2n-1)=n^2 \qquad (1)$$

We will prove that the statement $P(n)$ is true for all n.

We will check the above mentioned two properties (i) and (ii).

Observe that $P(n)$ is true for $n=1$, i.e.

$$P(1) = 1^2 = 1.$$

Now, we assume that $P(n)$ is true for $n=k$, i.e.

$$1+3+5+\ldots+(2k-1) = k^2 \qquad (2)$$

Adding $2k+1$ on both sides of (2) yields:

$$1+3+5+\ldots+(2k-1)+(2k+1) = k^2+2k+1$$

$$1+3+5+\ldots+(2k+1) = (k+1)^2$$

, which is true for $n=k+1$ (observe (1))

Therefore, $P(n)$ is true for all n, with $n \geq 1$.

Example:

Let $P(n)$ be the following statement:

$$1+2+3+\dots+n=\frac{n(n+1)}{2} \quad (1).$$

First we check that the statement is true for $n=1$,

$n=1$: LHS is equal to 1.

$n=1$: RHS is $\frac{1(1+1)}{2}=1$.

So, the statement is true for $n=1$ or $P(1)$ is true.

Suppose the statement is true for n and show that the statement is true for $n+1$:

$$1+2+3+\dots+n=\frac{n(n+1)}{2} \quad (2)$$

Adding $n+1$ on both sides yields

$$1+2+3+\dots+n+n+1=\frac{n(n+1)}{2}+n+1$$

$$=\frac{n^2+n+2n+2}{2}$$

$$=\frac{n^2+3n+2}{2}$$

$$=\frac{n^2+2n+n+2}{2}$$

$$=\frac{n(n+2)+(n+2)}{2}=\frac{(n+1)(n+2)}{2}$$

, which is true for $n+1$.

Therefore, the statement $P(n)$ is true for all $n \in \mathbb{N}$, with $n \geq 1$.

Example:

Show that the statement $P(n)$ given by:

$$P(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}.$$

holds for all $n \in \mathbb{N}$, with $n \geq 1$.

i) $n=1$:

$$LHS = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$RHS = \frac{1}{1+1} = \frac{1}{2}.$$

So, the statement $P(n)$ is true for $n=1$ ($P(1)$ is true)

ii) Now, let us suppose the statement $P(n)$ is true for n, i.e.:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1} \quad (1).$$

We will show that $P(n+1)$ is true, i.e.

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

By adding $\frac{1}{(n+1)(n+2)}$ on both sides of (1) we get:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

$$=\frac{n(n+2)+1}{(n+1)(n+2)}$$

$$=\frac{n^2+2n+1}{(n+1)(n+2)}$$

$$=\frac{(n+1)^2}{(n+1)(n+2)}$$

$$=\frac{n+1}{n+2}.$$

Therefore, the statement (1) is true for $n+1$.

Hence, $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 1$.

Example:

Show that $4^{2n}-1$ is a multiple of 15, for all $n \in \mathbb{N}$.

Proof:

Let $P(n)$ be the statement that:

$\exists\, x \in \mathbb{Z} : \quad 4^{2n}-1 = 15\cdot x \qquad (1)$

i) $n=1$: $\quad 4^{2\cdot 1}-1 = 16-1 = 15$,
which is true for $x=1$, since $15\cdot 1 = 15$.

ii) Let us suppose that the statement is true for $n=k$, i.e.

$$4^{2k}-1 = 15x, \text{ for some } x \in \mathbb{Z} \quad (2)$$

Now,
$$\begin{aligned}
4^{2\cdot(k+1)}-1 &= 4^{2k+2}-1 \\
&= 16\cdot 4^{2k}-1 \\
&= 15\cdot 4^{2k} + 4^{2k}-1 \\
&\overset{(1)}{=} 15\cdot 4^{2k} + 15\cdot x \\
&\overset{(1)}{=} 15\cdot(15x+1) + 15\cdot x \\
&= 15\,(15x+1+x) \\
&= 15\,(16x+1),
\end{aligned}$$

which is a multiple of 15, as $16x+1 \in \mathbb{Z}$ given that $x \in \mathbb{Z}$.

Example:

The sum of the cubes of three consecutive non-negative integers is divisible by 9.

Let $P(n)$ be the statement that

$$\exists\, x \in \mathbb{Z} : \quad n^3 + (n+1)^3 + (n+2)^3 = 9 \cdot x .$$

Then, $P(n)$ holds for all $n \in \mathbb{N}$.

Indeed,

i) $\underline{n=1}$: $P(1) = 1^3 + (1+1)^3 + (1+2)^3 = 1 + 8 + 27$
$= 36$
$= 9 \cdot 4$

So, $P(1)$ is true.

ii) Let us suppose that $P(n)$ holds for $n=k$,
i.e.

$$k^3 + (k+1)^3 + (k+2)^3 = 9 \cdot x, \quad x \in \mathbb{Z}. \quad (1)$$

We need to show that $P(n)$ holds for $n = k+1$,
i.e.

$$(k+1)^3 + \left[(k+1)+1\right]^3 + \left[(k+1)+2\right]^3 = 9 \cdot x , \quad x \in \mathbb{Z}.$$

Indeed,

$$(k+1)^3 + [(k+1)+1]^3 + [(k+1)+2]^3 = (k+1)^3 + (k+2)^3 + (k+3)^3$$

$$\overset{1}{=} 9x - k^3 + (k+3)^3$$

$$= 9x - \cancel{k^3} + 9k^2 + 27k + 27 - \cancel{k^3}$$

$$= 9x + 9k^2 + 27k + 27$$

$$= 9(x + k^2 + 3k + 3),$$

, which is true for $n = k+1$, as $n + k^2 + 3k + 3 \in \mathbb{Z}$.

Questions :

Practice

Show the following: (for $n \in \mathbb{N}, n \geq 1$)

a) $2+6+18+\ldots+2\cdot 3^{n-1} = 3^n - 1$

b) $1\cdot 3 + 2\cdot 5 + 3\cdot 7 + \ldots + n(2n+1) = \frac{n(n+1)(4n+5)}{6}$

c) $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

d) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

e) n^2+n is divisible by 2, $n \in \mathbb{N}$.

f) $8\cdot 10^n - 2$ is divisible by 6, $n \in \mathbb{N}$.

g) $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1}\cdot n^2 = \frac{(-1)^{n-1}\cdot n(n+1)}{2}$