Solutions for Tutorial 5 – Part 2 (Questions 6,7,8 and 9)

Question 6. State whether the following are true and explain why. If this is true, this can be done by arguing the proposition is true or demonstrating the negation is false. If it is false, then it can be done by demonstrating the proposition is false or that the negation is true.

Assume that $n \in Z$ and x, y, ε, $\delta \in \mathbb{R}$. If a condition is written after ∀ or ∃, this implies that a smaller set is taken, for example ∀ ε > 0 means for all positive (and real) ε.

(a) $\forall\, x, \exists\, y, 3x^2 - 2y = 5$.

The equality $3x^2 - 2y = 5$ is equivalent to $y = \frac{1}{2}(3x^2 - 5)$. Therefore, for any x, there is a y such that $3x^2 - 2y = 5$, namely $y = \frac{1}{2}(3x^2 - 5)$. So, this statement is TRUE.

(b) $\exists\, x, \forall\, y, 3x^2 - 2y = 5$.

Again, the equality $3x^2 - 2y = 5$ is equivalent to $y = \frac{1}{2}(3x^2 - 5)$. However, the quantifiers are different. For this to be true, there would have to be a value of x such that $y = \frac{1}{2}(3x^2 - 5)$ holds both for $y = 0$ and $y = 1$. This would give us $0 = \frac{1}{2}(3x^2 - 5) = 1$, which is not possible. So, this statement is FALSE.

(c) $\forall\, x, \exists\, y, 3x^2 - 2y^2 = 5$.

This statement is False. The equality $3x^2 - 2y^2 = 5$ is equivalent to $y^2 = \frac{1}{2}(3x^2 - 5)$. So, if $x = 1$ we would require that $y^2 = -1$, which is impossible (for real y). Thus, the statement is FALSE.

(d) $\exists\, x, \forall\, y, 3x^2 - 2y^2 = 5$.

Changing the quantifiers does not somehow make the statement true. Similarly to Part (b), we are asking for a value of x such that $3x^2 - 2y^2 = 5$ holds for both $y = 0$ and $y = 1$, which is false. Thus, this statement is also FALSE.

(e) $\forall\, \varepsilon > 0, \exists\, n, 2 -n < \varepsilon$.

This statement is TRUE. By taking logarithms of both sides, we see that the inequality is equivalent to $-n < log_2\varepsilon = (ln\varepsilon)/(ln2)$, which holds if and only if $n > -log_2\varepsilon$ (For ε close to 0, we note that $log_2\varepsilon$ is large and negative, and so $-log_2\varepsilon$ is large and positive.)

It is a standard fact that for any real number x there is an integer n such that $n > x$.

Question 7. Write down the negation of the following propositional functions and state whether the statement or its negation is true. You can assume n is an integer and x and y are real numbers.

(a) $\forall\, n, (n \leq 10) \wedge (n > 0)$.

We have $\neg(\forall\, n, (n \leq 10) \wedge (n > 0)) \equiv \exists\, n, (n > 10) \vee (n \leq 0)$.

The negation is true: there exists an integer which is greater than 10 or less than or equal to 0, for example 11, 12, 0, −1, ...

(b) $\exists\, n, (n \neq 0) \rightarrow (n^2 > 0)$.

We have $\neg(\exists\, n, (n \neq 0) \to (n^2 > 0)) \equiv \forall\, n, (n \neq 0) \wedge (n^2 \leq 0)$. Recall that the negation of an implication $p \to q$ is $p \wedge \neg q$. The original implication is true as the negation is false, if n is not zero then n^2 cannot be 0 or negative.

(c) $\exists\, x, \exists\, y, (y = 3x - 10) \wedge (y = -x + 2)$.

We have

$\neg(\exists\, x, \exists\, y, (y = 3x - 10) \wedge (y = -x + 2))$

$\equiv \forall\, x, \neg(\exists\, y, (y = 3x - 10) \wedge (y = -x + 2))$

$\equiv \forall\, x, \forall y, (y \neq 3x + 10) \vee (y \neq -x + 2))$.

First, we try to investigate whether the original statement is true or false. Any potential pair (x, y) exhibiting the truth of this statement must satisfy $y = 3x - 10$ and $y = -x + 2$ simultaneously. Solving these equations gives us $3x - 10 = -x + 2 (= y)$ and thus $4x = 12$, and so we get $x = 3$ (and also $y = -1$). Therefore, if both conditions $x = 3$ and $y = -1$ are satisfied, then original statement is true.

(d) $\forall\, x, \exists\, y, x^2 + y^2 = 1$.

We have $\neg(\forall\, x, \exists\, y, x^2 + y^2 = 1) \equiv \exists\, x, \forall\, y, x^2 + y^2 \neq 1$. The negation is true. Here we pick $x = 2$ (any x such that $|x| > 1$ will do) and note that for all y we have

$x^2 + y^2 = 4 + y^2 > 4$.

In particular, $x^2 + y^2$ can never be equal to 1 for that particular value of x. Therefore, the original statement is true.

Question 8. Let A , B be subsets of some universal set U. Using logical arguments, prove that $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

Proof:

$$
\begin{aligned}
A \subseteq B &\Leftrightarrow \forall\, x, x \in A \to x \in B \\
&\Leftrightarrow \forall\, x, x \notin A \vee x \in B \\
&\Leftrightarrow \forall\, x, x \in B \vee x \notin A \\
&\Leftrightarrow \forall\, x, x \notin B^c \vee x \in A^c \\
&\Leftrightarrow \forall\, x, x \in B^c \to x \in A^c \\
&\Leftrightarrow B^c \subseteq A^c .
\end{aligned}
$$

Thus, $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

Please note that in the second and second last steps, we have used the equivalent propositions

$\neg\, p \vee q$ and $p \to q$.

In the third step, we have used the commutativity of $\vee$. In the fourth step, we have used the concept of the compliment of a set.

Question 9. $\wp(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.