Naive Set Theory

PART 2

Proving some of the Properties

Commutative Property

$$\boxed{A \cup B = B \cup A}$$

$$x \in A \cup B \iff (x \in A) \vee (x \in B)$$

$$\xLeftrightarrow{(\because \vee \text{ is commutative})} (x \in B) \vee (x \in A)$$

$$\iff x \in B \cup A.$$

Therefore, $\boxed{A \cup B = B \cup A}$

Distributive Property

$$\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$$

$$x \in A \cup (B \cap C) \iff (x \in A) \vee (x \in (B \cap C))$$

$$\iff (x \in A) \vee (x \in B \wedge x \in C)$$

(Distributivity of $\vee$ over $\wedge$)

$$\iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$$

$$\iff (x \in A \cup B) \wedge (x \in A \cup C)$$

$$\iff x \in (A \cup B) \cap (A \cup C).$$

So, $\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$.

Similarly, we can prove:

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$$

Homework

Some Properties of Set Difference

$$x \in A \setminus B \iff (x \in A) \wedge (x \notin B)$$

For sets A, B, C, we have:

(i) $A \setminus A = \emptyset$

(ii) $A \setminus B = \emptyset \iff A \subseteq B$

(iii) In general, $A \setminus B \neq B \setminus A$.
However, for a special case when

$$A \setminus B = B \setminus A \iff A = B.$$

(iv) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(v) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

(vi) $A \setminus B = A \setminus (A \cap B)$

Proof of (iv):

$$x \in A \setminus (B \cup C) \iff (x \in A) \wedge (x \notin B \cup C)$$

$$\iff (x \in A) \wedge \neg (x \in B \cup C)$$

$$\iff (x \in A) \wedge \neg (x \in B \vee x \in C)$$

$$\iff (x \in A) \wedge (\neg (x \in B) \wedge \neg (x \in C))$$

$\because (p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r))$

$$\iff ((x \in A) \wedge \neg (x \in B)) \wedge ((x \in A) \wedge \neg (x \in C))$$

$$\iff (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$$

$$\iff (x \in A \setminus B) \wedge (x \in A \setminus C)$$

$$\iff x \in [(A \setminus B) \cap (A \setminus C)]$$

$\blacksquare$

Disjoint Sets

As we have discussed before, two sets are disjoint if $A \cap B = \emptyset$.

- A and $\emptyset$ are disjoint for any set A.
- $A \setminus B$ and $B \setminus A$ are disjoint for any sets A and B.

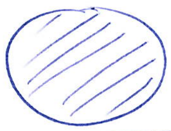
Can you show these?

The Venn Diagram

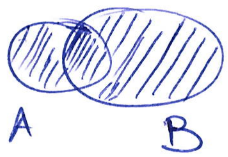
Operations on sets can easily be understood using **shaded pictures**, known as Venn diagrams.

Some Examples:

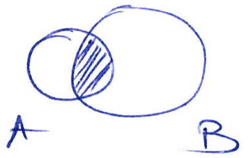
A



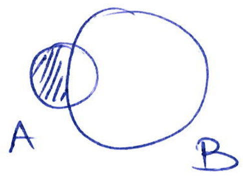
$A \cup B$



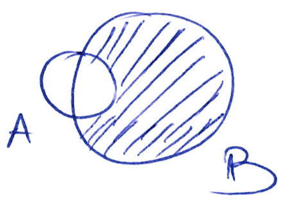
$A \cap B$

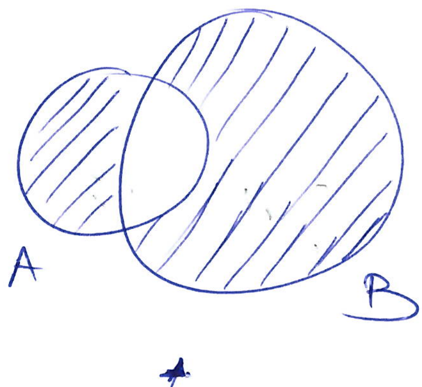


$A \setminus B$



$B \setminus A$





$(A \setminus B) \cup (B \setminus A)$

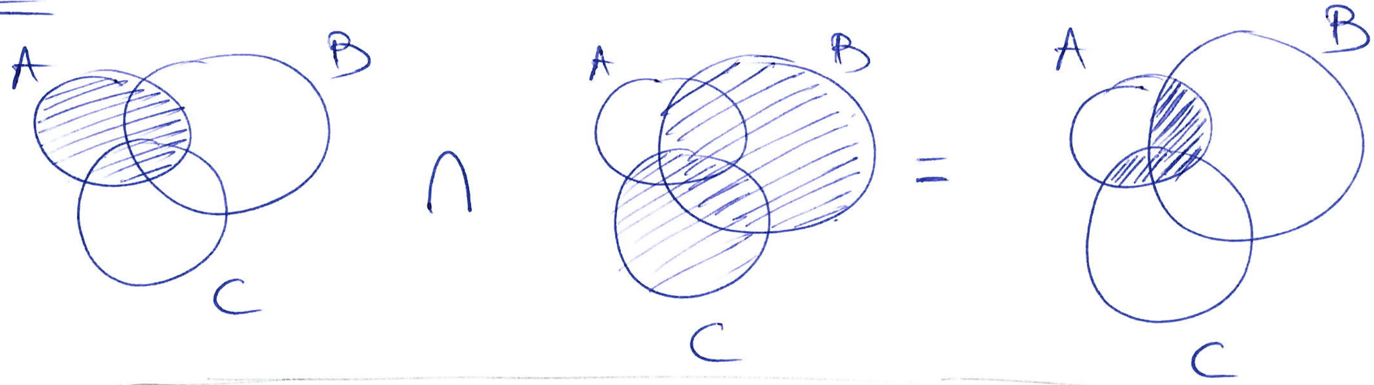
or

$(A \cup B) \setminus (A \cap B)$

Examples:

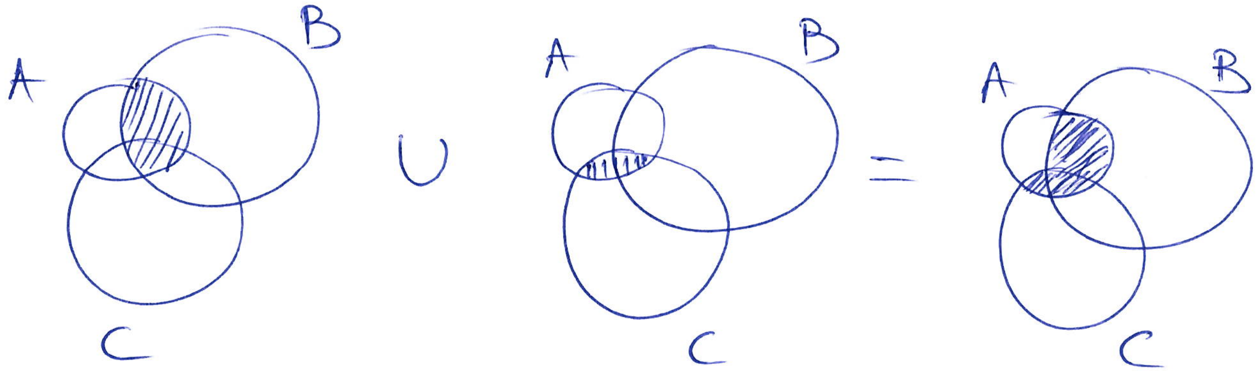
1) Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using Venn diagrams.

LHS:



$$\boxed{A \quad \cap \quad B \cup C \quad = A \cap (B \cup C)}$$

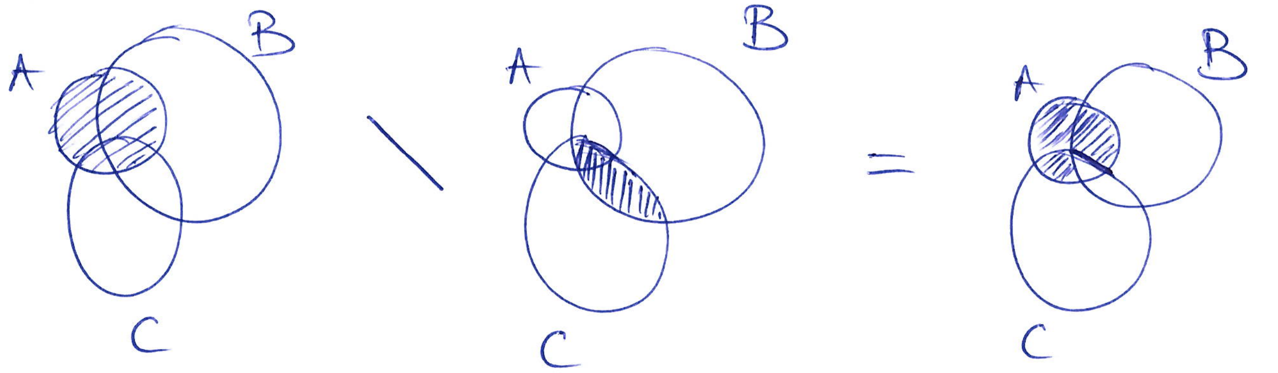
RHS:



$$\boxed{A \cap B \quad \cup \quad A \cap C \quad = (A \cap B) \cup (A \cap C)}$$

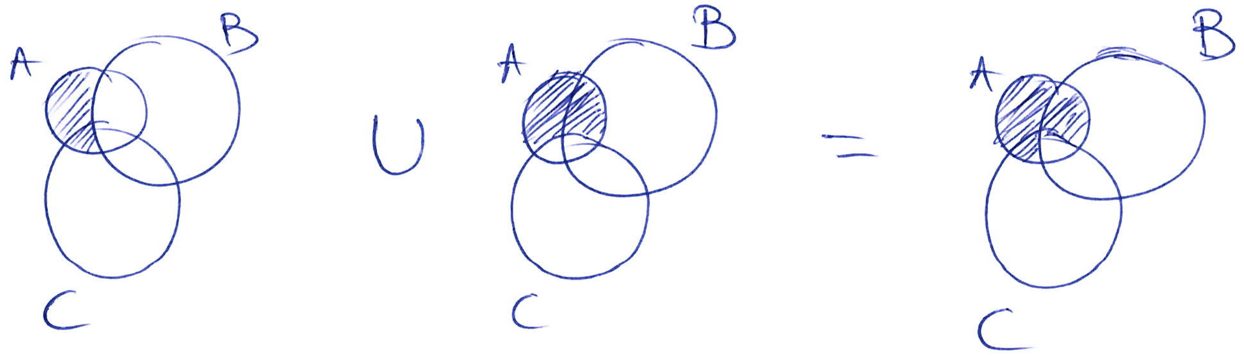
2) Prove $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ by using Venn diagrams.

<u>LHS:</u>



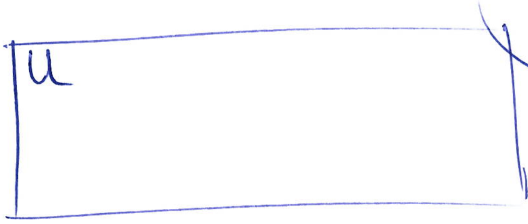
$A \quad \setminus \quad B \cap C \quad = \quad A \setminus B \cap C$

<u>RHS:</u>



$A \setminus B \quad \cup \quad A \setminus C \quad = (A \setminus B) \cup (A \setminus C)$

For a universal set $\mathcal{U}$ or U, normally a rectangle / box is used for a Venn diagram.



Therefore,

$A^c = U \setminus A$

Some important rules for the complement of a set:

(i) $A^{cc} = A$

(ii) $(A \cup B)^c = A^c \cap B^c$

(iii) $(A \cap B)^c = A^c \cup B^c$

(ii) and (iii): De Morgan Laws

(iv) $A \cap A^c = \emptyset$, $A \cup A^c = U$

Proof of (ii):

for $x \in U$, $x \in (A \cup B)^c \iff x \notin A \cup B$

$$\iff \neg (x \in A \cup B)$$

$$\iff \neg (x \in A \lor x \in B)$$

$$\iff \neg (x \in A) \land \neg (x \in B)$$

$$\iff x \notin A \land x \notin B$$

$$\iff x \in A^c \land x \in B^c$$

$$\iff x \in A^c \cap B^c$$

$\blacksquare$

Some more properties:

- $(A \cup B^c)^c = A^c \cap B$,
- $(A \cap B^c)^c = A^c \cup B$,
- $(A^c \cup B^c)^c = A \cap B$,
- $(A^c \cap B^c)^c = A \cup B$.

Power Set:

The Power Set of a set S is the set of all subsets of S and is denoted by $\mathcal{P}(S)$.

Examples:

$A = \{1, 2\}$, $B = \{a, b, c\}$, $C = \phi$.

$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$\mathcal{P}(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$\mathcal{P}(\phi) = \{\phi\}$

Cardinality of a set:

Number of elements in a set S is said to be the cardinality of the set S and is denoted by $|S|$.

Remark: If $|S| = n$, then $|\mathcal{P}(S)| = 2^n$.