Mathematical Logic

In mathematical algorithms or Theorems, we need to use some <u>Logical expressions</u>, such as:

If p, then q

If p and q, then r and s.

Therefore, it is important to know when these expressions are TRUE or FALSE. In fact, the "truth value" of such expressions is required.

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The fundamental notion in logic is that of "proposition".

Proposition

A proposition is a declarative sentence (statement) that is either True or False (but not both).
The truth or falsehood of a proposition is called its truth value

Examples

(i) Ice floats in water.

(ii) $2+3=5$

(iii) France is in Asia.

(iv) $8<3$

(v) Where do you live?

(vi) What is your name?

(vii) This sentence is false.

(viii) x is an even number.

(ix) Daniel

Statement

Compound

Many propositions are composite, i.e. they are composed of subpropositions and various connectives (these will be defined subsequently).
Such composite propositions are called compound propositions.

Primitive

A proposition is said to be primitive, if it cannot be broken into simpler propositions

Examples

p: David is a student.

q: England is a university.

r: Roses are red and grass is green.

s: Tom is tall or he sleeps late at night.

t: P_1 is rational.

Basic Logical Operations

(Connectives and Truth Tables)

There are three basic logical operations

Conjunction , Disjunction , Negation.

They, respectively, correspond to:

Conjunction $\longrightarrow$ AND

Disjunction $\longrightarrow$ OR

Negation $\longrightarrow$ NOT

Name	Represented	"Meaning"
Conjunction	$p \wedge q$	"p and q"
Disjunction	$p \vee q$	"p or q (or both)"
Negation	$\neg p$	"not p"

Conjunction

Two propositions can be combined by the word "and"

in order to form a compound proposition, which is known as the conjunction of the two original propositions.

Symbolically,

"$p \wedge q$" reads "p and q"

and denotes the conjunction of p and q.

Is $p \wedge q$ a proposition? ....

Therefore, it has a truth value, which depends on the truth values of ... and ....

In order to examine the truth value of $p \wedge q$ we construct the following truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Examples

p: Ice floats in water. (T)

q: $2+3=5$ (T)

$p \wedge q$: Ice floats in water and $2+3=5$ (T)

r: $2+3=6$ (F)

$p \wedge r$: Ice floats in water and $2+3=6$ (F)

Disjunction

Two propositions can be combined by the word "or" to form a compound proposition which is called a disjunction of the original proposition.

"$p \vee q$" reads as "p or q" denoting the disjunction of p and q.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

"If p and q are false, then $p \vee q$ is false. Otherwise, $p \vee q$ is true."

Examples

p: Ice floats in water (T)

q: $2+3=5$ (T)

r: $2+3=6$ (F)

s: $3>9$ (F)

$p \vee q$: Ice floats in water or $2+3$ makes 5 (T)

$p \vee r$: Ice floats in water or $2+3=6$ (T)

$s \vee q$: $3>9$ or $2+3=5$ (T)

(3 is greater than 9)

$s \vee r$: $3>9$ or $2+3=6$ (F)

Negation

Given any proposition p, we can form the negation of p ($\neg p$) by writing

" it is <u>not true</u> that ..p.. " or

" it is <u>false</u> that ..p.. " or

by simply inserting in p the word "not".

Symbolically,

the negation of p, reads as "not p" is denoted by "$\neg p$".

The truth value of $\neg p$ depends on the truth value of p.

p	$\neg p$
T	F
F	T

"If p is true, then $\neg p$ is false".

and

"If p is false, then $\neg p$ is true".

Examples

p: Ice floats in water. (T)

$\neg p$: not(Ice floats in water). (F)

→ Ice does not float in water.

q: $3 > 9$ (3 is greater than 9) (F)

$\neg q$: not (3 is greater than 9)

3 is not greater than 9 (T).

$3 \not> 9$

$\neg(3 > 9)$

$3 \leq 9$

Propositions and Truth Tables

Let $S(p, q, \dots)$ denote an expression constructed from logical variables $p, q, \dots$ which take on value true (T) or false (F) and the logical connectives $\wedge, \vee, \neg$.

Such an expression $S(p, q, \dots)$ will be called a proposition.

Truth tables are very important to understand the properties of a proposition.

Let us construct the truth table for the proposition $\neg(p \wedge \neg q)$.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T.

Equivalence of Propositions

Two propositions are equivalent, if they have some truth table and we write $p \equiv q$.

Examples

$$\left.\begin{aligned} \neg(p \wedge q) &\equiv (\neg p) \vee (\neg q) \\ \neg(p \vee q) &\equiv (\neg p) \wedge (\neg q) \end{aligned}\right\} \text{De Morgan's Law}$$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Tautology

Tautology is a statement which is always true.

Example:

The proposition "p or not p", i.e. $p \vee \neg p$ is a tautology.

WHY?

See the truth table below

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Contradictions

A contradiction is a proposition which is always false, e.g. $p \wedge \neg p$

(Can you show this?)

More Propositions

Implication

Some statements in Mathematics are of the following form "If p, then q". These statements are called
conditional statements
and are denoted by

$$p \longrightarrow q.$$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

If I win a lottery, then I will buy you a present.

If I win a lottery and I do not buy you a present, then I am lying.

I can buy a present and still tell the truth without winning a lottery.

NOTE!

$p \to q$

is <u>not true</u> (false) only if the implication is violated.

If the hypothesis is not satisfied, then the implication is not violated no matter what truth value q has.

Consider the difference between

mathematical logic	and	everyday language
p implies q. simply means that if p is true, then so is q.	VS.	We say p implies q. We think p causes q (is causing) to happen.

Note:

Implications

- if P, then Q
- Q if P
- P implies Q
- Q is implied by P
- P only if Q
- P is a sufficient condition for Q
- Q is a necessary condition for P

Equivalence

$$p \longleftrightarrow q$$

(p if and only if q)

iff.

$(p \to q) \wedge (q \to p)$

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T.

Homework

$$p \to q \equiv \neg p \vee q$$

Example:

"If I win a lottery, I will buy you a present".

"Either I do not win the lottery or I will buy you a present"

Both above statements are logically equivalent, as they contain the same information.

Algebra of propositions

Associative laws

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Commutative laws

$$p \vee q \equiv q \vee p$$

$$p \wedge q \equiv q \wedge p$$

Distributive laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Involution law

$$\neg\neg p = p$$

- Can you check these laws?
- You can verify these laws by constructing their truth table.

Inverse

The inverse of a conditional proposition $p \to q$ is the proposition $\neg p \to \neg q$.

NB. See Question 6 in this week's tutorial sheet.

Converse

The converse of a conditional proposition $p \to q$ is the proposition $q \to p$.

Example:

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Contrapositive

The contrapositive of a conditional proposition $p \to q$ is the proposition $\neg q \to \neg p$.

They are logically equivalent.

Quantifiers (Two special constructions)

These two special constructions enable us to generalise a statement which involves variable(s).

"m is odd". (depends on m)

"m is odd, for all integers m"

(FALSE)

In this case m is a dummy variable.

* Therefore, we can make use of it so that it ranges over a set of values and then ... forget it.

Note:

"For every integer m, m is odd".*

or "Every integer is odd".

*: not the best way to write it ☺

Best way: " m is odd, for all integers m".

"For all" can be expressed as implications.

e.g. "If m is an integer, then m is odd".

"Let ...."

e.g. Let k be an integer. Then, k is odd.

"m is odd for some integer"

or

"There is an odd integer".

(There exists)