# Formative assessment 10

Question A 1

$$\begin{cases} \dot{y}_1 = -\frac{1}{2} y_1 + \frac{5}{2} y_2 \\ \dot{y}_2 = \frac{5}{2} y_1 - \frac{1}{2} y_2 \end{cases} \iff \dot{Y} = AY \quad \text{where } Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -1/2 & 5/2 \\ 5/2 & -1/2 \end{pmatrix}$$

(a) Find the general solution.

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

where $u_1$, $u_2$ are the eigenvectors of $A$ corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$ respectively.

Let us calculate the eigenvalues and eigenvectors of $A$.

Eigenvalues

$$\det(A - \lambda Id) = \det \begin{pmatrix} -\frac{1}{2} - \lambda & 5/2 \\ \frac{5}{2} & -\frac{1}{2} - \lambda \end{pmatrix} = \left(-\frac{1}{2} - \lambda\right)^2 - \frac{25}{4} = 0$$

$$\left(\frac{1}{2} + \lambda\right)^2 = \frac{25}{4}$$

$$\left(+\frac{1}{2}+\lambda\right)^2 = \left(\frac{5}{2}\right)^2$$

$$\lambda + \frac{1}{2} = \pm\frac{5}{2}$$

$$\lambda_1 = -\frac{5}{2} - \frac{1}{2} = -3 \qquad \boxed{\lambda_1 = -3}$$

$$\lambda_2 = \frac{5}{2} - \frac{1}{2} = 2 \qquad \boxed{\lambda_2 = 2}$$

Eigenvectors of A

- $\lambda_1 = -3$ Solving the eigen value problem

$$A u_1 = \lambda_1 u_1 \qquad \text{where } u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & 5/2 \\ 5/2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = -3 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{2}p_1 + \frac{5}{2}q_1 = -3p_1 \\ \frac{5}{2}p_1 - \frac{1}{2}q_1 = -3q_1 \end{cases} \qquad \begin{cases} -p_1 + 5q_1 = -6p_1 \\ 5p_1 - q_1 = -6q_1 \end{cases} \qquad \begin{cases} 5p_1 + 5q_1 = 0 \\ 5p_1 + 5q_1 = 0 \end{cases}$$

$$p_1 + q_1 = 0$$

We choose $p_1 = 1 \Rightarrow q_1 = -1 \qquad u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

• $\lambda_2 = 2$ Solving the eigenvalue problem.

$$A u_2 = \lambda_2 u_2 \qquad \text{where} \quad u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

$$\begin{pmatrix} -1/2 & 5/2 \\ 5/2 & -1/2 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}$$

$$\begin{cases} -\frac{1}{2} p_2 + \frac{5}{2} q_2 = 2 p_2 \\ \frac{5}{2} p_2 - \frac{1}{2} q_2 = 2 q_2 \end{cases} \qquad \begin{cases} -p_2 + 5q_2 = 4 p_2 \\ 5p_2 - q_2 = 4 q_2 \end{cases} \qquad \begin{cases} -5p_2 + 5q_2 = 0 \\ 5p_2 - 5q_2 = 0 \end{cases}$$

$$p_2 - q_2 = 0$$

We choose $p_2 = 1$ $\quad q_2 = 1$ $\quad \boxed{u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

The general solution is

$$y(t) = D_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + D_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} y_1(t) = D_1 e^{-3t} + D_2 e^{2t} \\ y_2(t) = -D_1 e^{-3t} + D_2 e^{2t} \end{cases}$$

where $D_1, D_2$ are arbitrary constant.

What type of fixed point is $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ?

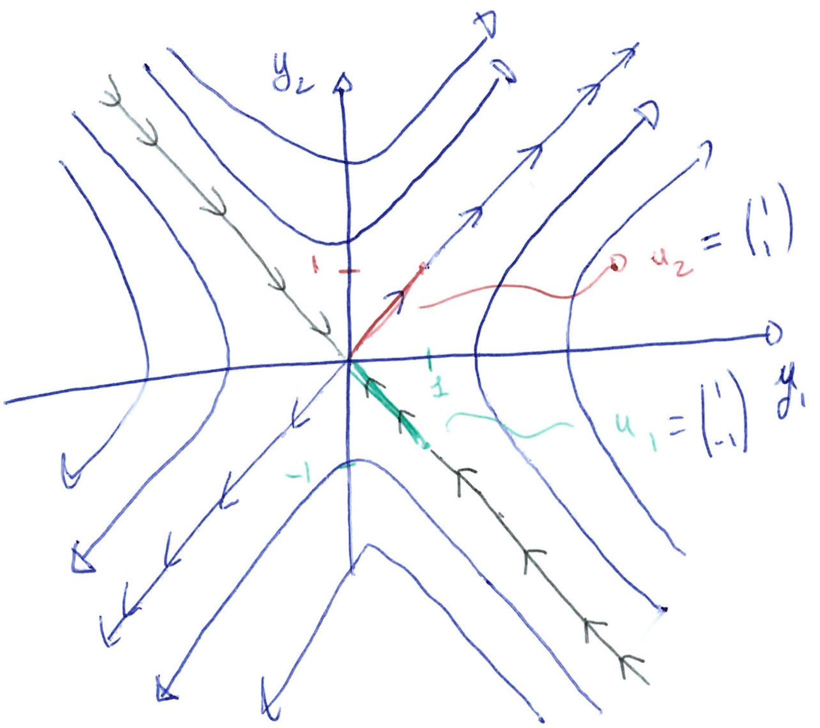
Since $\lambda_1 = -3 < 0$ and $\lambda_2 = 2 > 0$ the fixed point is a

SADDLE

Schetch the phase portrait.

Since $\lambda_1 = -3 < 0$ the eigenvector $u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ defines the direction of the STABLE INVARIANT MANIFOLD

Since $\lambda_2 = 2 > 0$ the eigenvector $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ defines the direction of the UNSTABLE INVARIANT MANIFOLD.

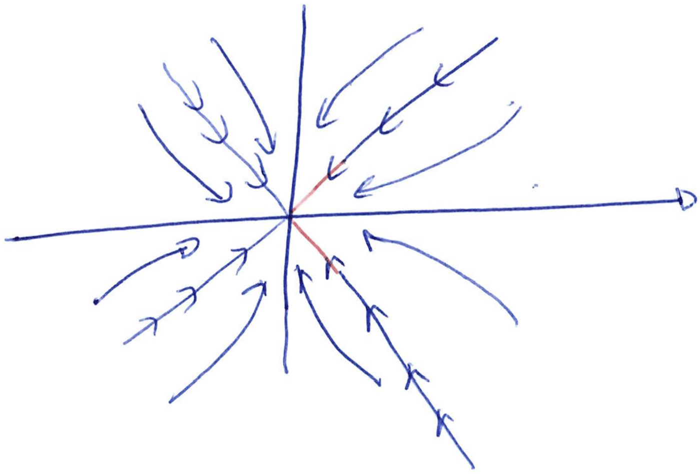



The phase portrait of the dynamical system $\square$.

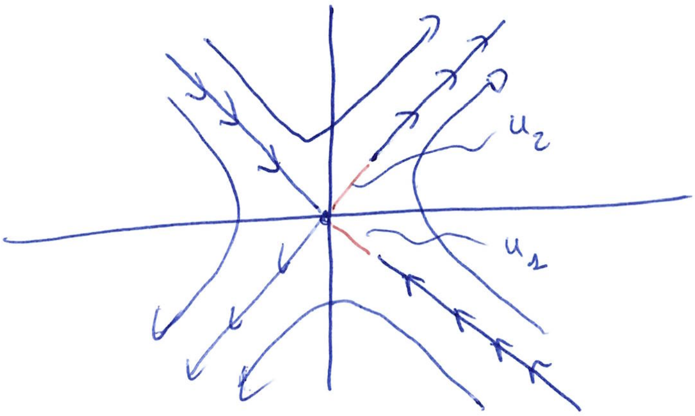
# Phase portrait of a linear system

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad \lambda_1 \neq \lambda_2$

ASYMPTOTICALLY STABLE • STABLE NODE $\lambda_1 < 0 \;\; \lambda_2 < 0$

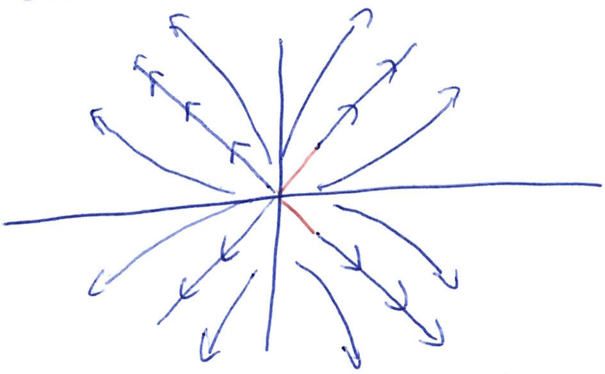


UNSTABLE • SADDLE $\lambda_1 < 0 \;\; \lambda_2 > 0$




A saddle is an unstable fixed point.

UNSTABLE • UNSTABLE NODE $\lambda_1 > 0 \;\; \lambda_2 > 0$

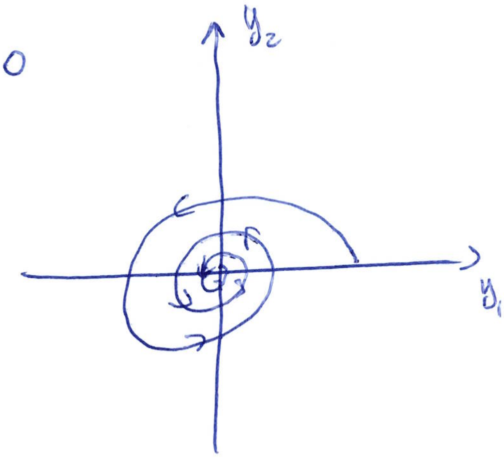


$\lambda_1, \lambda_2$ complex conjugate

ASYMPTOTIC STABLE

STABLE FOCUS

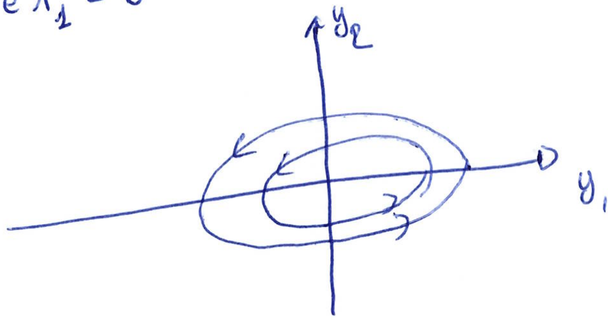
$Re\lambda_1 < 0$




STABLE

CENTRE

$Re\lambda_1 = 0$




UNSTABLE

UNSTABLE FOCUS

$Re\lambda_1 > 0$