Exam 2018 - Question 2

Consider the differential equation

$$\left(1 - by \sin x\right) + 2\cos x \quad y' = 0$$

(a) find the value of the parameter b for which the given ODE is exact.

$P(x,y) + Q(x,y)\, y' = 0$ is exact

iff. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

In our case $P(x,y) = 1 - by \sin x$ $\quad Q(x,y) = 2\cos x$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(1 - by\sin x\right) = -b\sin x$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(2\cos x\right) = -2\sin x$$

$$-b\sin x = -2\sin x \iff \boxed{b = 2}$$

The ODE is exect iff $b=2$ $\square$

(b) For the parameter b found in (a) above, find the solution which satisfies the initial condition $y(0)=1$.

The exact ODE has implicit solution

$$F(x,y(x)) = C \quad \text{with } \underline{P(x,y)=\frac{\partial F}{\partial x}}, \quad \underline{Q(x,y)=\frac{\partial F}{\partial y}}.$$

Therefore

$$F = \int P(x,y)\,dx = \int (1-2y\sin x)\,dx =$$

$$= x + 2y\cos x + g(y)$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(x + 2y\cos x + g(y)\right) = 2\cos x + g'(y) = Q(x,y) = 2\cos x$$

$$2\cos x + g'(y) = 2\cos x$$

$$\Rightarrow g'(y) = 0 \qquad \Rightarrow g(y) = C'$$

Therefore

$$F(x,y) = x + 2y\cos x + C'$$

The implicit general solution is $\quad F(x,y) = C$

$$x + 2y\cos x = C - C'$$

Imposing the I.C. $y(0)=1$ $\qquad x=0,\ y=1$

$$\underbrace{0+2\cdot 1\cos 0}_{=1} = C-C' \implies C-C'=2$$

The solution to the IVP is $\qquad 2y\cos x = 2-x$

$$x+2y\cos x = 2 \implies y=\frac{2-x}{2(\cos x)}$$

The explicit solution is

$$y=\frac{2-x}{2(\cos x)}$$

(c) Consider the initial value problem.

$$\frac{dy}{dx}=f(x,y), \quad f(x,y)=\sqrt{3y^2+16}, \quad \text{I.C. } y(1)=0$$

Show that the Picard-Lindelöf theorem ensures the uniqueness and existence of a solution to the above problem in a rectangular region $|x-a|\le A$, $|y-b|\le B$ and specify the parameters a and b. Write down the maximal value of A for $B=4$.

The Picard - Lindelöf theorem establishes the existence and uniqueness of the solution in a rectangular region D centered at the initial condition $y(a) = b$.

D: $|x-a| \leq A$ $\quad |y-b| \leq B$ $\quad$ where I.C. is $y(a) = b$.

For us I.C. $y(1) = 0$ $\Rightarrow$ $a = 1$ $\quad b = 0$

D: $|x-1| \leq A$ $\quad |y| \leq B$.

The hypotheses of the Picard - Lindelöf theorem are:

(1) $f(x,y)$ is continuous in D. $\exists A > 0, B > 0$ such that $f(x,y)$ is continuous in D.

In our case $f(x,y) = \sqrt{3y^2 + 16}$ $\quad$ D: $|x-1| \leq A$
$|y| \leq B$

Yes! This function is continuous in D for any choice of $A > 0$, $B > 0$

(2) $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\sqrt{3y^2+16} = \dfrac{1}{2\cancel{x}\sqrt{3y^2+16}}\, 6\,\cancel{6}^{3}y$

$\dfrac{\partial f}{\partial y}$ should be bounded in D.

- This is true because $\dfrac{\partial f}{\partial y}$ is continuous in D for any choice of $A>0$, $B>0$ ✓

- Check directly

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{6y}{2\sqrt{3y^2+16}}\right| = \left|\frac{3y}{\sqrt{3y^2+16}}\right| < \frac{3\theta}{\sqrt{3}} \quad ✓$$

bounded in D.

(3) $A \le \dfrac{B}{M}$ where $M = \max\limits_{D} |f(x,y)| = \max\limits_{D} \left|\sqrt{3y^2+16}\right|$

$D:\ |x-1| \le A \ , \ |y| \le B$

$$M = \max_{-B\le y\le B} \sqrt{3y^2+16} = \sqrt{3B^2+16}$$

$$A \leq \frac{B}{M} = \frac{B}{\sqrt{3B^2+16}}$$

To find the maximal A for B = 4

$$\frac{A}{M} = \left.\frac{B}{\sqrt{3B^2+16}}\right|_{B=4} = \frac{4}{\sqrt{3 \cdot 16 + 16}} = \frac{4}{\sqrt{4 \cdot 16}} = \frac{1}{2}$$

$$\frac{A}{M} = \frac{1}{2}$$

$$\left| \frac{3y}{\sqrt{3y^2+16}} \right| < \frac{3}{\sqrt{3}} \iff 3|y| < \frac{3}{\sqrt{3}} \sqrt{3y^2+16}$$

$$\frac{3}{\sqrt{3}} \sqrt{3y^2+16} > \frac{3}{\sqrt{3}} \sqrt{3y^2} = \frac{3\sqrt{3}\,|y|}{\sqrt{3}}$$

Summary of the module

Weeks 1-3 First-order ODEs Picard-Lindelöf theorem

Weeks 4-6 Second order ODEs the theorem of the Alternative.

Weeks 8-11 Linear systems of ODEs.

Week 8 - Equilibria
linearisation of non-linear systems

Weeks 9-10 Linear system of ODEs
Fixed points
Phase portraits.

Week 11 Stability of linear and non-linear systems of ODEs
Lyapunov function.