Structure of the 2022-2023 exam.

(80% of the final mark)

4 Questions (Each including several points)

Question 1-3 cover the first part of the module

Question 4 covers the second half of the module.

All questions will be handwritten.

You can train for the exam with

a) formative assessments

b) past papers.

Formative assessment week 5

1) Solve the ODE

$$y'' + 7y' + 6y = 10 \sin 2x \qquad (1)$$

This is an inhomogeneous ODE

The corresponding homogeneous ODE is

$$y'' + 7y' + 6y = 0 \qquad (2)$$

whose characteristic equation is

$$M_2(\lambda) = \lambda^2 + 7\lambda + 6 = 0$$

of roots $\lambda = \dfrac{-7 \pm \sqrt{49 - 24}}{2} = \dfrac{-7 \pm \sqrt{25}}{2} = \dfrac{-7 \pm 5}{2} = \begin{cases} -6 \\ -1 \end{cases}$

$\lambda_1 = -6 \quad \lambda_2 = -1$

The general solution to the homogeneous ODE (2) is

$$y_h(x) = c_1 e^{-6x} + c_2 e^{-x}$$

where c_1, c_2 are arbitrary real constants.

Since the ODE (1) is of the type

$a = 2$

$$a_2 y'' + a_1 y' + a_0 y = p(x) \sin a x$$

$p(x) = 10$

$(k=0)$

as long as $p(x)$ is a polynomial of degree k and $a \neq \lambda_1$, $a \neq \lambda_2$

we can use the educated guess method.

Therefore we look for a solution of the type

$$y_p(x) = Q(x) \left[A' \cos 2x + B' \cos 2x \right]$$

where A', B' are ~~arbitrary~~ constant that need to be determined and

$Q(x) = d_0$ is a polynomial of degree $k=0$ whose

coefficient d_0 needs to be determined.

$$y_p(x) = A \cos 2x + B \overset{\sin}{\cancel{\cos}} 2x \qquad \text{where } A = d_0 A', \; B = d_0 B'$$

Let us substitute $y_p(x)$ into the ODE (1). To this end let us calculate

$$y_p'(x) = -2A \sin 2x + 2B \overset{\cos}{\cancel{\sin}} 2x$$

$$y_p''(x) = -4A \underset{\cos}{\overset{\cancel{\sin}}{\cancel{\cos}}} 2x - 4B \sin 2x$$

Inserting $y_p'(x)$, $y_p''(x)$, $y_p(x)$ into (1) ~~we~~ get.

Given (1) is $y'' + 7y' + 6y = 10 \sin 2x$

$$\left[-4A \cos 2x - 4B \sin 2x\right] + 7\left[-2A \sin 2x + 2B \cos 2x\right] +$$
$$+ 6\left[A \cos 2x + B \sin 2x\right] = 10 \sin 2x$$

$$\cos 2x \left(-4A + 14B + 6A\right) + \sin 2x \left(-4B - 14A + 6B\right) =$$
$$= 10 \sin 2x$$

$$\begin{cases} -4A + 14B + 6A = 0 \\ -4B - 14A + 6B = 10 \end{cases} \qquad \begin{cases} 2A + 14B = 0 \\ 2B - 14A = 10 \end{cases} \Rightarrow \begin{cases} A = -7/10 \\ B = 1/10 \end{cases}$$

Therefore the solution to the inhomogeneous ODE (1)

$$y(x) = y_h(x) + y_p(x) \qquad \text{with}$$

$$y_h(x) = c_1 e^{-6x} + c_2 e^{-x} \qquad \text{with } c_1, c_2 \text{ arbitrary constants}$$

$$y_p(x) = -\frac{7}{10} \cos 2x + \frac{1}{10} \sin 2x$$

$$y(x) = C_1 e^{-5x} + C_2 e^{-x} - \frac{7}{10}\cos 2x + \frac{1}{10}\sin 2x.$$

$\square$