# Lyapunov function

We consider a general system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix}$$

having an equilibrium point at $Y_*(t) = \begin{pmatrix} a \\ b \end{pmatrix}$

A Lyapunov function $V(y_1, y_2)$ is a continuously differentiable function defined for $|Y(t) - Y_*(t)| < R$ such that.

1. $V(a,b) = 0$
2. $V(y_1, y_2) > 0$ for $(y_1, y_2) \neq (a,b)$
3. $D_f(V) \leq 0$ for $(y_1, y_2) \neq (a,b)$

If a Lypunov function exists than $Y_*(t)$ is a Lyapunov stable solution.

If the Lypunov function satisfies.

3.' $D_f(V) < 0$ for $(y_1, y_2) \neq (a,b)$

then $Y_*(t)$ is asymptotically stable.

<u>Example</u> Verify that $V = y_1^2 + y_2^2$ is a valid Lyapunov function of the system of ODEs around the equilibrium point $Y_*(t) = (0,0)^T$.

$$\begin{cases} \dot{y}_1 = -y_2 - y_1^3 \\ \dot{y}_2 = y_1 - y_2^3 \end{cases} \qquad (*)$$

Is the zero solution asymptotically stable?

<u>Solution</u> ① Check that $Y_*(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point of (*)

$$\dot{y}_1 = -y_2 - y_1^3 = f_1(y_1, y_2)$$

$$\dot{y}_2 = y_1 - y_2^3 = f_2(y_1, y_2)$$

$Y_*(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point because

$$f_1(0,0) = 0 \quad \checkmark$$

$$f_2(0,0) = 0 \quad \checkmark$$

② Let us check that $V(y_1, y_2) = y_1^2 + y_2^2$ is a valid Lyapunov function.

$\boxed{1}$ $V(0,0) = y_1^2 + y_2^2 \Big|_{(y_1,y_2)=(0,0)} = 0$ ✓

$\boxed{2}$ $V(y_1,y_2) = y_1^2 + y_2^2 > 0$ if $(y_1,y_2) \neq (0,0)$ ✓

$\boxed{3}$ $$D_f(V) = \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2$$

$$= \frac{\partial V}{\partial y_1} f_1(y_1,y_2) + \frac{\partial V}{\partial y_2} f_2(y_1,y_2)$$

$V = y_1^2 + y_2^2$

$\frac{\partial V}{\partial y_1} = 2y_1$ $\frac{\partial V}{\partial y_2} = 2y_2$ $f_1(y_1,y_2) = -y_2 - y_1^3$

$f_2(y_1,y_2) = y_1 - y_2^3$

Therefore

$$D_f(V) = 2y_1\left[-y_2 - y_1^3\right] + 2y_2\left[y_1 - y_2^3\right] =$$

$$= \cancel{-2y_1y_2} - 2y_1^4 + \cancel{2y_1y_2} - 2y_2^4 = -2(y_1^4 + y_2^4)$$

$D_f(V) = -2(y_1^4 + y_2^4) < 0$ for $(y_1,y_2) \neq (0,0)$

(3) V is a Lyapunov function for *

The zero solution is ASYMPTOTICALLY STABLE.

If the system of ODEs is of the type

$$\begin{cases} \dot{y}_1 = -\dfrac{\partial V}{\partial y_1} \\ \dot{y}_2 = -\dfrac{\partial V}{\partial y_2} \end{cases}$$

then the dynamics is called gradient flow

$V$ is the Lyapunov function around the equilibrium point

$$y^*(t) = (a, b) \quad \text{where} \quad \left(-\frac{\partial V}{\partial y_1}, -\frac{\partial V}{\partial y_2}\right)\Bigg|_{(y_1,y_2)=(a,b)} = (0,0)$$

Further topics on the stability of ODEs.

What conclusions can we draw about the stability of an equilibrium point of a non-linear system of ODE based on the stability of its linearized system?

<u>Theorem</u> Consider the following system of ODEs having $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as an equilibrium point

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \text{higher-order terms.} \qquad (1)$$

where $A$ is a $2 \times 2$ time independent matrix having eigenvalues $\lambda_1, \lambda_2$.

The linearised system of (1) around the zero solution

$$\dot{Y} = AY \qquad (2)$$

1. If $Re\,\lambda_1 < 0$ $Re\,\lambda_2 < 0$ $\quad s = \max(Re\,\lambda_1, Re\,\lambda_2) < 0$

then the zero solution is ASYMPTOTICALLY STABLE for system (1)

2. If $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) > 0$ then

the zero solution is UNSTABLE

and system (1)

3. If $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) = 0$ then

the zero solution of the system (1) can be

either stable or unstable.

The stability needs to be established by considering

the non-linear terms in the system (1).

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Therefore

- If the linearised system is ASYMPTOTICALLY STABLE

  the non-linear system is also ASYMPTOTICALLY STABLE

- If the linearised system is UNSTABLE

  the non-linear system is also UNSTABLE

- If the linearised system is STABLE but not asymptotically stable

  the so called linear stability argument fails and in order to establish the stability of the solution it is necessary to ~~look~~ & consider the non-linear terms.

# Summary of the second half of the module

Week 8 Dynamical systems, equilibrium points
Linearisation of non-linear systems.

Week 9-10 We have solved linear system of ODEs with constant coefficients. Phase portraits.

Week 11 Stability of solutions. Relation between the stability of equilibrium point of non-linear systems and their corresponding linearised system.