Stability criteria for linear systems of ODEs with constant coefficients.

We consider the linear system

$$\dot{Y} = AY \qquad (1)$$

where A is a 2×2 matrix with constant coefficients and eigenvalues λ_1, λ_2 $\lambda_1 \neq \lambda_2$

This system admits the zero solution $Y(t) = (0,0)^T$ or otherwise $(y_1, y_2) = (0,0)$ is an equilibrium point.

<u>Theorem</u> Define $s = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2)$ then the zero solution $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is

1. UNSTABLE if $s > 0$
2. STABLE (Lyapunov stable) if $s = 0$
3. ASYMPTOTICALLY STABLE if $s < 0$

Indeed $Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium point of (1)

The general solution to (1) for any initial condition

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

(1) $\lambda_1, \lambda_2 \in \mathbb{R}$ $\qquad D_1, D_2 \in \mathbb{R}$

- If $s = \max(\lambda_1, \lambda_2) > 0$ $\qquad e^{\lambda_1 t} \to \infty$ or $e^{\lambda_2 t} \to \infty$ as $t \to \infty$

The trajectory goes to infinity

The zero solution is UNSTABLE

SADDLE or UNSTABLE NODE

- If $s = \max(\lambda_1, \lambda_2) < 0$ $\qquad e^{\lambda_1 t} \to 0$ and $e^{\lambda_2 t} \to 0$ as $t \to \infty$

The zero solution is ASYMPTOTICALLY STABLE

STABLE NODE.

- If $s = \max(\lambda_1, \lambda_2) = 0$ $\qquad$ If $\lambda_1 = 0$ and $\lambda_2 < 0$

The zero solution is STABLE. $\qquad e^{\lambda_1 t} = 1 \quad e^{\lambda_2 t} \to 0$ as $t \to \infty$

② λ_1, λ_2 complex conjugate $\quad \lambda_1 = \alpha + i\beta \quad \alpha, \beta \in \mathbb{R} \quad \beta > 0$

$\lambda_2 = \alpha - i\beta$

$S = \max(\mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2) = \alpha \qquad e^{\lambda_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} \cdot e^{i\beta t}$

The general solution reads

$$Y(t) = e^{\alpha t} \underbrace{\left(D_1 e^{i\beta t} u_1 + D_2 e^{-i\beta t} u_2 \right)}_{\text{oscillatory part.}}$$

- If $s = \alpha > 0 \qquad e^{\alpha t} \to \infty$ as $t \to \infty$

  The zero solution is USTABLE

  UNSTABLE FOCUS

- If $s = \alpha < 0 \qquad e^{\alpha t} \to 0$ as $t \to \infty$

  The zero solution is ASYMPTOTICALLY STABLE

  STABLE FOCUS.

- If $s = \alpha = 0 \qquad e^{\alpha t} = 1$

  The zero solution is STABLE

  CENTRE.

③ These results can be extended to the case $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

- If $\lambda > 0$ the zero solution is UNSTABLE
- If $\lambda < 0$ the zero solution is ~~STAB~~ ASYMPTOTICALLY STABLE
- If $\lambda = 0$ the zero solution is STABLE.

What can we say about the stability of the equilibrium point for a non-linear system of ODEs?

Lyapunov function method for investigating the stability of non-linear systems of ODEs

We consider the system of ODES

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix} \qquad (2)$$

Suppose that $Y(t)$ is a solution to (2) with $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

For any continuously differentiable function $V(\tilde{y}_1, \tilde{y}_2)$ we can define its value at a given time t along the solution $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ given by $v(t) = V(y_1(t), y_2(t))$

The directional derivative of V along $\vec{f} = (f_1(t, y_1, y_2), f_2(t, y_1, y_2))$

or orbital derivative

$$D_f(V) = \dot{v} = \frac{\partial V}{\partial y_1} \dot{y}_1 + \frac{\partial V}{\partial y_2} \dot{y}_2$$

Using the system of ODE (2)

$$\dot{y}_1 = f_1(t, y_1, y_2)$$
$$\dot{y}_2 = f_2(t, y_1, y_2)$$

we have

$$D_f(V) = \frac{\partial V}{\partial y_1} f_1(t, y_1, y_2) + \frac{\partial V}{\partial y_2} f_2(t, y_1, y_2)$$

Theorem : Lyapunov stability theorem.

Let $Y_*(t) = (a, b)^T$ be an equilibrium solution of (2)

Assume that inside the circle $0 < |Y(t) - Y_*(t)| < R$ there exist a continuously differentiable function $V(y_1, y_2)$ satisfying

1. $V(a, b) = 0$
2. $V(y_1, y_2) > 0$ for any $(y_1, y_2) \neq (a, b)$
3. $D_f(V) \leq 0$ for any $(y_1, y_2) \neq (a, b)$

$V(y_1, y_2)$ is called the Lyapunov function of (2)

The solution $Y_*(t) = (a, b)^T$ is Lyapunov stable

Theorem Lyapunov asymptotically stability theorem

If the Lyapunov function satisfies 1. 2. and 3.' where

3.' $D_f(V) < 0$ for any $(y_1, y_2) \neq (a, b)$

Then the solution $Y(t) = (a, b)^T$ is ASYMPTOTICALLY STABLE.