Stability of solutions of ODEs

Question Consider a system of ODEs and a given solution $Y_*(t)$ to the IVP with initial condition $Y_*(0)$

How different will be a solution $Y(t)$ to the same system of ODEs with a slightly different I.C. $Y(0)$?

Will the two solutions $Y(t)$ and $Y_*(t)$

- converge to each other
- remain close
- or diverge

in the limit $t \to \infty$?

In order to address this question we need to first formalize the notion of stability of a solution of a general (linear and non-linear) system of ODEs.

We will define

1. Lypunov stability
2. Asymptotic stability.

We consider a general system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix} \qquad (1)$$

which includes also autonomous systems when the functions f_1 and f_2 do not depend explicitely on time, i.e.

$$f_1(t, y_1, y_2) = f_1(y_1, y_2)$$
$$f_2(t, y_1, y_2) = f_2(y_1, y_2)$$

Includes linear and non-linear systems.

Lyapunov stability

The concept

" $Y_*(t)$ is Lyapunov stable if any solution $Y(t)$ remains close to $Y_*(t)$ as long as the two initial conditions $Y_*(0)$ and $Y(0)$ are sufficiently close to each other "

Lyapunov stability

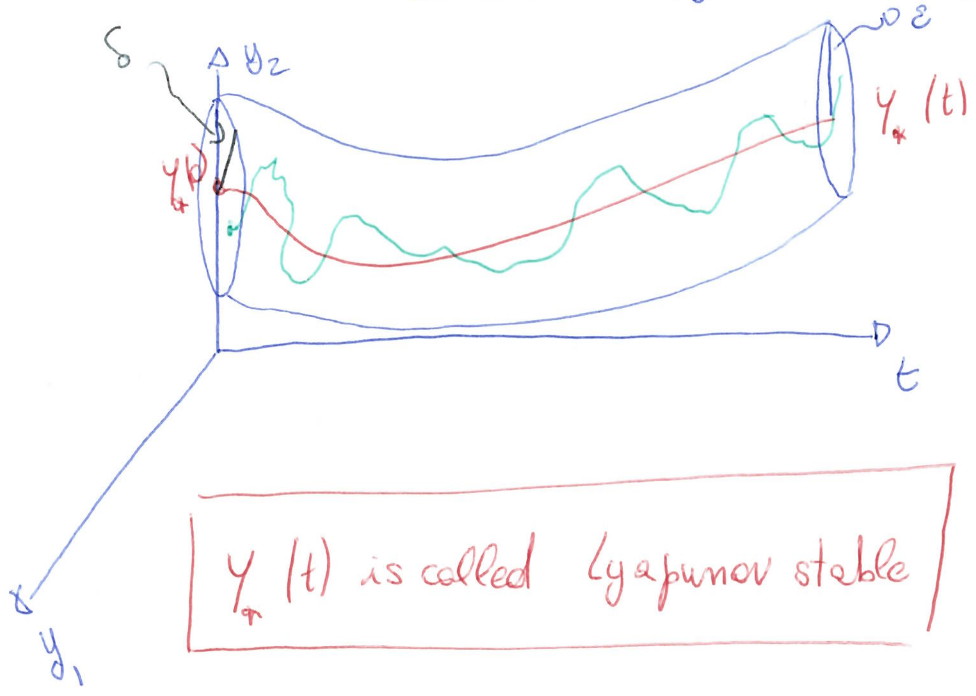
Definition A solution $Y_*(t)$ of the system of ODEs (1) corresponding to the I.C. $Y_*(0) = (a_1, b_1)^T$ is Lyapunov stable if for any arbitrary $\varepsilon > 0$ we can find a $\delta > 0$ such that

if another initial condition $Y(0) = (a_2, b_2)^T$ is chosen inside a circle of radius δ centered in $Y_*(0)$, i.e.

if $|Y(0) - Y_*(0)| < \delta$

then for any time $t > 0$ the solution $Y(t)$ corresponding to the I.C. $Y(0)$

1. exists
2. will satisfy $|Y(t) - Y_*(t)| < \varepsilon \quad \forall t$



The solution will remain inside the "tube" of radius ε around the solution $Y_*(t)$

$$\boxed{Y_*(t) \text{ is called Lyapunov stable}}$$

Lyapunov stability (continuation)

In mathematical terms we can express the definition of Lyapunov stability as

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \qquad \text{if } |Y(0) - Y_*(0)| < \delta$$

$$\text{then } |Y(t) - Y_*(t)| < \varepsilon \quad \forall t.$$

Note Given $Y_*(0) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $Y(0) = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

we can express $|Y(0) - Y_*(0)| < \delta$ as $|Y(0) - Y_*(0)|^2 < \delta^2$

$$\Rightarrow (a_1 - a_2)^2 + (b_1 - b_2)^2 < \delta^2$$

Moreover given $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ and $Y_*(t) = \begin{pmatrix} y_1^*(t) \\ y_2^*(t) \end{pmatrix}$

We can express $|Y(t) - Y_*(t)| < \varepsilon$ as $|Y(t) - Y_*(t)|^2 < \varepsilon^2$

$$\Rightarrow (y_1(t) - y_1^*(t))^2 + (y_2(t) - y_2^*(t))^2 < \varepsilon^2 \qquad \forall t$$

Asymptotic stability

Definition

The solution $Y_\alpha(t)$ of the system of ODEs (1) with IC. $Y_\alpha(0) = (a_1, b_1)^T$ is called ASYMPTOTICALLY STABLE if

1. it is Lyapunov stable
2. there exist a $\delta > 0$ such that if

$$|Y(0) - Y_\alpha(0)| < \delta \quad \text{then}$$

$$|Y(t) - Y_\alpha(t)| \to 0 \quad \text{as } t \to \infty$$

Lyapunov stability does not always imply asymptotic stability.

Example Is the zero solution $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the system

$$\dot{Y} = AY \quad \text{with } A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

Lyapunov stable?

Is it asymptotically stable?

Solution A has eigenvalues and eigenvectors

$$\lambda_1 = 2i \qquad\qquad u_1 = \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$\lambda_2 = -2i \qquad\qquad u_2 = \begin{pmatrix} 2 \\ i \end{pmatrix}$$

The general solution of the system of ODEs is

$$Y = D_1 e^{2it} u_1 + D_2 e^{-2it} u_2$$

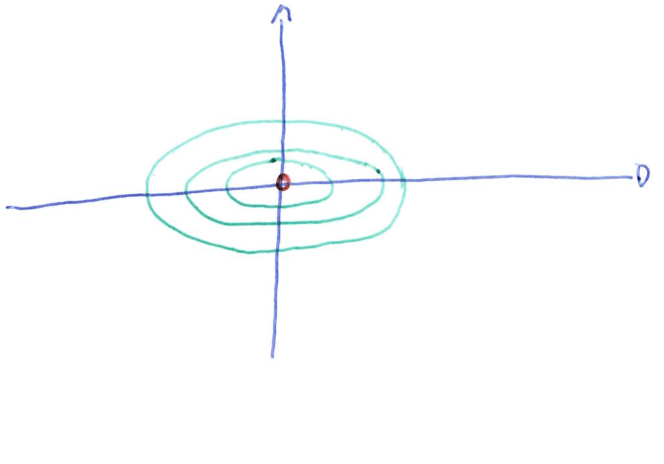
where D_1 and D_2 are complex conjugate.

The solution to the IVP with I.C. $Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$ is

$$\begin{cases} y_1(t) = a \cos 2t - 2b \sin 2t \\ y_2(t) = \frac{a}{2} \sin 2t + b \cos 2t \end{cases}$$

with $\quad y_1^2 + 4 y_2^2 = a^2 + 4 b^2 \quad$ (see tutorial week 11)

This trajectory is an ellipse



$Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution

$Y(0,0)$ is Lyapunov stable but not asymptotically stable.

Check the stability of the solution $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ such that if $|Y(0) - Y_*(0)| < \delta$

then $|Y(t) - Y_*(t)| < \varepsilon$

Notice the $Y_*(t) = Y_*(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $Y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$

therefore we need to check

$\forall \varepsilon > 0 \quad \exists \delta > 0$ such that if $|Y(0)| < \delta$ then $|Y(t)| < \varepsilon$

Assume $|Y(0)| < \delta \Rightarrow |Y(0)|^2 < \delta^2 \Rightarrow (a^2 + b^2) < \delta^2$

Let us calculate explicitelly $|Y(t)|^2$

$$|Y(t)|^2 = y_1^2(t) + y_2^2(t) \underset{\substack{\uparrow \\ \text{since } y_2^2(t) \geq 0}}{\leq} y_1^2(t) + 4y_2^2(t) \underset{\uparrow}{=} a^2 + 4b^2$$

because

$y_1^2 + 4y_2^2 = a^2 + 4b^2$

is the trajectory.

Therefore

$$|Y(t)|^2 \leq a^2 + 4b^2 \underset{\substack{\uparrow \\ a^2 \geq 0}}{\leq} 4a^2 + 4b^2 = 4(a^2 + b^2) < 4\delta^2$$

If we choose $\varepsilon = \delta/2$ $\Rightarrow$ $|Y(t)|^2 < \varepsilon^2$

$\Rightarrow$ $|Y(t)| < \varepsilon$

The zero solution is Lyapunov stable.

The zero solution is NOT asymptotically stable.

$\square$