Phase portraits : case of complex eigenvalues.

We consider the linear system of ODEs

$$\dot{Y} = AY$$

Where A is a real 2x2 matrix with complex eigenvalues

$$\lambda_1 = \alpha + i\beta \qquad \alpha, \beta \in \mathbb{R} \qquad \beta > 0$$
$$\lambda_2 = \alpha - i\beta$$

and eigenvectors u_1 & u_2 (which are also complex conjugate)

The general solution is given by

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2 \qquad \text{Eq. (1)}$$

In order for Y to be real, D_1 and D_2 must be complex conjugate

$$D_1 = \frac{1}{2}(\tilde{a} + i\tilde{b})$$
$$D_2 = \frac{1}{2}(\tilde{a} - i\tilde{b})$$

where $\tilde{a}, \tilde{b} \in \mathbb{R}$ arbitrary constants determined by the initial condition.

We consider the basis of vectors v_1, v_2 defined as

$$v_1 = \mathrm{Re}\, u_1 \qquad\qquad v_2 = -\,\mathrm{Im}\, u_1$$

so that
$$\begin{cases} u_1 = v_1 - i v_2 \\ u_2 = v_1 + i v_2 \end{cases}$$

v_1, v_2 are linearly independent.

The general solution $Y = Y(t)$ in this v_1, v_2 basis can be written as

$$Y = \tilde{y}_1 v_1 + \tilde{y}_2 v_2$$

with
$$\begin{cases} \tilde{y}_1(t) = e^{\alpha t}\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right) \\ \tilde{y}_2(t) = e^{\alpha t}\left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right) \end{cases} \qquad \text{Eq. (1)}$$

$\square$

Proof at the end

Where $\tilde{a}, \tilde{b} \in \mathbb{R}$ arbitrary constants determined by the initial condition. Assuming the initial condition for $t = 0$

$$\begin{cases} \tilde{y}_1(0) = \tilde{a} \\ \tilde{y}_2(0) = \tilde{b} \end{cases}$$

What is the phase portrait of this dynamical system in the coordinates $\tilde{y}_1, \tilde{y}_2$?

Let consider

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}\left[\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right)^2 + \left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right)^2\right]$$

$$= e^{2\alpha t}\left[\tilde{a}^2\cos^2\beta t + \tilde{b}^2\sin^2\beta t \cancel{- 2\tilde{a}\tilde{b}\cos\beta t\sin\beta t} + \right.$$

$$\left. \tilde{a}^2 \sin^2\beta t + \tilde{b}^2\cos^2\beta t \cancel{+ 2\tilde{a}\tilde{b}\cos\beta t\sin\beta t}\right] =$$

Using $\cos^2\beta t + \sin^2\beta t = 1$

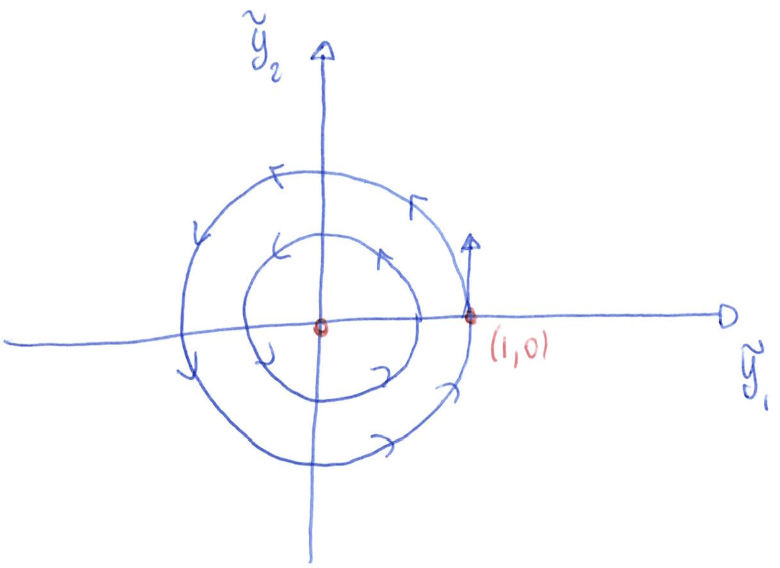
$$= e^{2\alpha t}\left[\tilde{a}^2\left(\cos^2\beta t + \sin^2\beta t\right) + \tilde{b}^2\left(\cos^2\beta t + \sin^2\beta t\right)\right]$$

$$= e^{2\alpha t}\left(\tilde{a}^2 + \tilde{b}^2\right)$$

$$\boxed{\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}\left(\tilde{a}^2 + \tilde{b}^2\right)}$$

These characterize the trajectories in the new coordinates $(\tilde{y}_1, \tilde{y}_2)$. These trajectories can be used to classify phase portrait depending on the value $\boxed{\alpha = \operatorname{Re}\lambda_1}$

① Case I $\boxed{\alpha = 0}$ $(0,0)$ is a CENTRE (neutral stability)



The trajectories satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = \tilde{a}^2 + \tilde{b}^2$$

"$y^2 + x^2 = R^2$"

The trajectories are circles in the coordinates $(\tilde{y}_1, \tilde{y}_2)$

We take $\tilde{y}_1(0) = \tilde{a} = 1$

$\tilde{y}_2(0) = \tilde{b} = 0$

The trajectory is counter-clockwise in the coordinates $\tilde{y}_1, \tilde{y}_2$

Arrows will be determined by the tangent vector $(\dot{\tilde{y}}_1(0), \dot{\tilde{y}}_2(0))$ at $t=0$

Let us take $\alpha = 0$, $\tilde{a} = 1$, $\tilde{b} = 0$

We know that

$$\tilde{y}_1(t) = (\tilde{a}\cos\beta t - \tilde{b}\sin\beta t) = \cos\beta t$$

$$\tilde{y}_2(t) = (\tilde{a}\sin\beta t + \tilde{b}\cos\beta t) = \sin\beta t$$

Deriving with respect to time

$$\begin{cases} \dot{\tilde{y}}_1 = -\beta \sin\beta t \\ \dot{\tilde{y}}_2 = \beta \cos\beta t \end{cases} \quad \text{At } t=0 \Rightarrow \begin{cases} \dot{y}_1(0) = 0 \\ \dot{y}_2(0) = \beta > 0 \end{cases}$$

Careful! This discussion is valid only in the coordinates $(\tilde{y}_1, \tilde{y}_2)$

In the original coordinates (y_1, y_2)

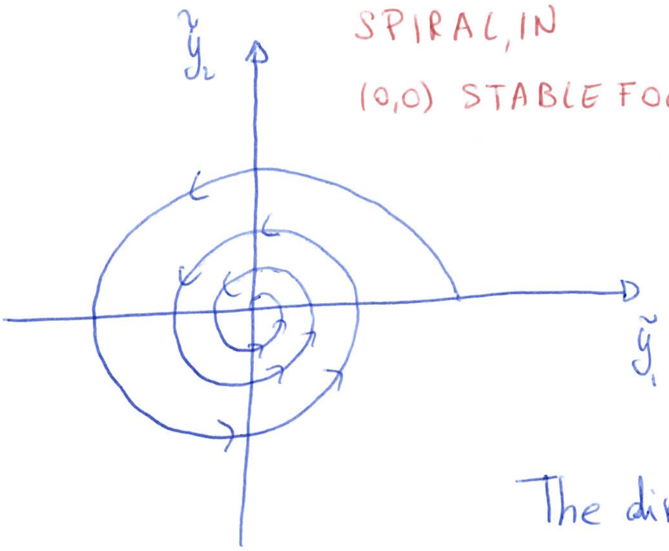
a) the trajectories can be circles or ellipses

b) the direction can be clockwise or counterclockwise.

Case II STABLE SPIRAL, STABLE FOCUS. $\boxed{\alpha < 0}$

The trajectories in the coordinates $\tilde{y}_1, \tilde{y}_2$ satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t}(\tilde{a}^2 + \tilde{b}^2)$$



For $t \to \infty$ $e^{2\alpha t} \to 0$ $(\alpha < 0)$

$\tilde{y}_1^2 + \tilde{y}_2^2 \to 0$ as $t \to \infty$

The radial coordinate goes to zero!

The direction in the coordinates $\tilde{y}_1, \tilde{y}_2$ is counterclockwise but in the original coordinates y_1, y_2 it can be clockwise or counterclockwise.

Case III UNSTABLE SPIRAL, (0,0) UNSTABLE FOCUS

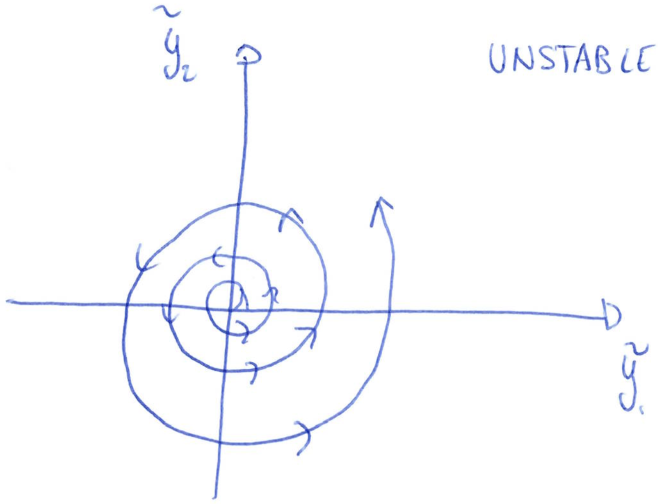
$\boxed{\alpha > 0}$

The trajectories in the coordinates $\tilde{y}_1, \tilde{y}_2$ satisfy

$$\tilde{y}_1^2 + \tilde{y}_2^2 = e^{2\alpha t} (\hat{a}^2 + \hat{b}^2)$$

As $t \to \infty$ $\quad e^{2\alpha t} \to \infty \quad (\alpha > 0)$

The trajectories go to infinity!



UNSTABLE SPIRAL, (0,0) UNSTABLE FOCUS.

The trajectories in the coordinates $\tilde{y}_1, \tilde{y}_2$ are counter-clockwise but in the original coordinates y_1, y_2 they can be clockwise or counterclockwise.

Derivation of Eq(2) from Eq.(1) EXTRA MATERIAL

Proof. $$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

Y is real $\qquad D_1 = \frac{1}{2}(\tilde{a} + i\tilde{b}) \qquad \lambda_1 = \alpha + i\beta$

$u_1 = V_1 - iV_2 \qquad D_2 = \frac{1}{2}(\tilde{a} - i\tilde{b}) \qquad \lambda_2 = \alpha + i\beta$

$u_2 = V_1 + iV_2$

$$Y = \frac{1}{2}(\tilde{a} + i\tilde{b}) e^{(\alpha + i\beta)t} (V_1 - iV_2) +$$

$$+ \frac{1}{2}(\tilde{a} - i\tilde{b}) e^{(\alpha \mp i\beta)t} (V_1 + iV_2)$$

Using $c_1 + \bar{c}_1 = 2 \operatorname{Re} c_1$ where $c_1 \in \mathbb{C}$

$$Y = 2 \operatorname{Re}\left[\frac{1}{2}(\tilde{a} + i\tilde{b}) e^{(\alpha + i\beta)t} (V_1 - iV_2)\right] =$$

Using $e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i \sin\beta t)$

$$Y = \operatorname{Re}\left[(\tilde{a} + i\tilde{b}) e^{\alpha t} (\cos\beta t + i \sin\beta t)(V_1 - iV_2)\right]$$

~~Using~~

$$= e^{\alpha t} \operatorname{Re}\left\{\left[(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t) + i(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t)\right](V_1 - iV_2)\right]$$

EXTRA MATERIA (continuation)

Using $Re(c_1 c_2) = Re\, c_1\, Re\, c_2 - Im\, c_1\, Im\, c_2$

$$Y = e^{\alpha t}\left[\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right)v_1 + \left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right)v_2\right]$$

Since $Y = \tilde{y}_1 v_1 + \tilde{y}_2 v_2$ with $\tilde{y}_1, \tilde{y}_2$ uniquely determined

it follows

$$\begin{cases} \tilde{y}_1 = e^{\alpha t}\left(\tilde{a}\cos\beta t - \tilde{b}\sin\beta t\right) \\ \tilde{y}_2 = e^{\alpha t}\left(\tilde{a}\sin\beta t + \tilde{b}\cos\beta t\right) \end{cases}$$