Phase portraits in the new coordinates

Consider the linear system of ODEs

$$\mathring{Y} = A\,Y$$

where A is a real 2×2 matrix with real and distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ corresponding to the two linearly independent eigenvectors u_1 and u_2

$$Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2 \qquad \text{with}$$

$$\tilde{Y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \qquad \text{following} \qquad \mathring{\tilde{Y}} = \tilde{A}\,\tilde{Y}$$

$$\text{where} \qquad \tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Hence we have

$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} \end{cases}$$

where D_1, D_2 are arbitrary real constants.

Let us now study the phase portrait in the new variables $\tilde{y}_1, \tilde{y}_2$!

We know that

$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} \end{cases} \qquad (1)$$

We make two observations:

(A) If $\tilde{y}_1(t=a) = 0$ then $D_1 = 0$

$$0 = \tilde{y}_1(t=a) = D_1 \underbrace{e^{\lambda_1 a}}_{\neq 0} \Rightarrow D_1 = 0$$

Therefore $\tilde{y}_1(t) = D_1 e^{\lambda_1 t} = 0 \quad \forall t$

$\Rightarrow$ This implies that $\tilde{y}_1 = 0$ is an INVARIANT MANIFOLD

Similarly $\tilde{y}_2 = 0$ is also an INVARIANT MANIFOLD

(B) If $D_1 \neq 0$ and $D_2 \neq 0$ Eq. (1) can be also written as

$$\frac{\tilde{y}_1}{D_1} = e^{\lambda_1 t} = \left(e^t\right)^{\lambda_1} \Rightarrow e^t = \left(\frac{\tilde{y}_1}{D_1}\right)^{1/\lambda_1} \quad \text{for } \lambda_1 \neq 0$$

$$\frac{\tilde{y}_2}{D_2} = e^{\lambda_2 t} = \left(e^t\right)^{\lambda_2} = \left[\left(\frac{\tilde{y}_1}{D_1}\right)^{1/\lambda_1}\right]^{\lambda_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\lambda_2/\lambda_1}$$

Therefore the trajectory is given by the eq.

$$\boxed{\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\frac{\lambda_2}{\lambda_1}}}$$

Trajectory $\quad D_1 \neq 0 \quad D_2 \neq 0$

$\lambda_1 \neq 0$

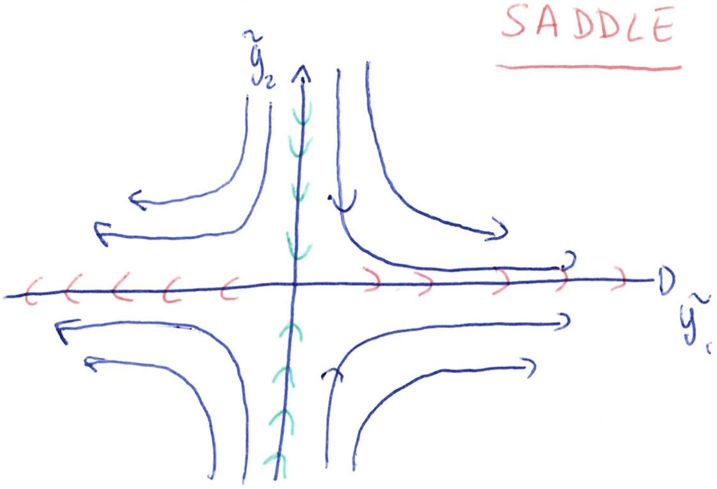
Which curves are those?

$\boxed{1}$ Case I $\qquad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2 \qquad \boxed{\lambda_1 > 0, \ \lambda_2 < 0}$

$$\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1}\right)^{\frac{\lambda_2}{\lambda_1}} = \left(\frac{\tilde{y}_1}{D_1}\right)^{-\left|\frac{\lambda_2}{\lambda_1}\right|}$$

- For $\left|\frac{\lambda_2}{\lambda_1}\right| = 1 \qquad$ "$y = x^{-1}$" $\qquad$ these are hyperbolas

SADDLE



$\tilde{y}_2 = 0 \quad (\lambda_1 > 0)$ UNSTABLE MANIFOLD

$\tilde{y}_1 = 0 \quad (\lambda_2 < 0)$ STABLE MANIFOLD

As $t \to \infty$

$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} & (\lambda_1 > 0) \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} & (\lambda_2 < 0) \end{cases}$$

$\tilde{y}_1 \to \operatorname{sign}(D_1) \cdot \infty$

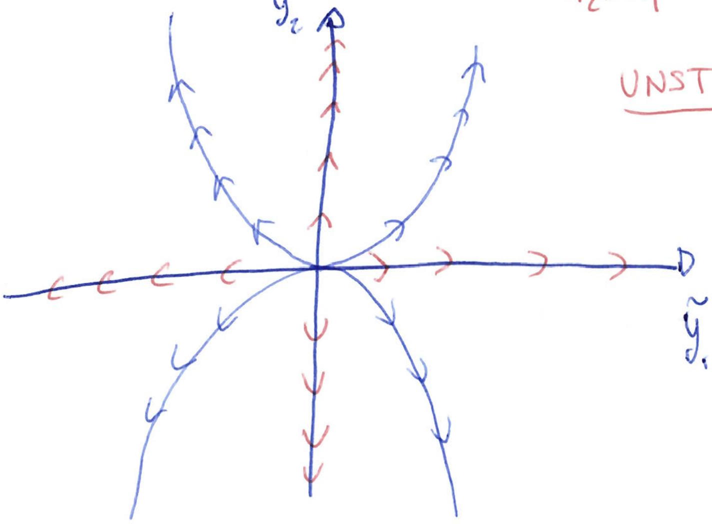
$\tilde{y}_2 \to 0$

$\boxed{2}$ Case II $\quad \lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1 \neq \lambda_2$ $\quad \boxed{\lambda_1 > 0 \quad \lambda_2 > 0}$

Trajectories
$$\frac{\tilde{y}_2}{D_2} = \left(\frac{\tilde{y}_1}{D_1} \right)^{\frac{\lambda_2}{\lambda_1}}$$

If $\frac{\lambda_2}{\lambda_1} = 2$ $\quad$ " $y = x^2$ " $\quad$ these are parabolas.

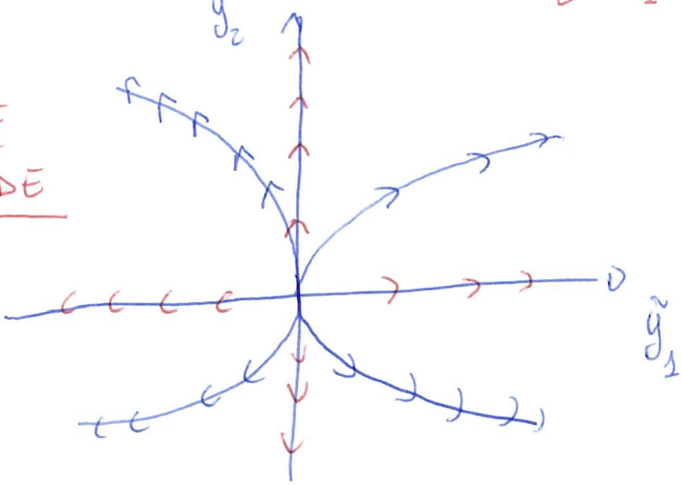
$\lambda_2 > \lambda_1$



UNSTABLE

NODE

$\lambda_2 < \lambda_1$



As $t \to \infty$

$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} \end{cases}$$

As $t \to \infty$

$\tilde{y}_1(t) \to (\text{sign } D_1)\, \infty$

$\tilde{y}_2(t) \to (\text{sign } D_2)\, \infty$

$\boxed{3}$ Case III $\quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad \lambda_1 \neq \lambda_2$ $\quad \boxed{\lambda_1 < 0 \quad \lambda_2 < 0}$

Trajectories
$$\frac{\tilde{y}_2}{D_1} = \left(\frac{\tilde{y}_1}{D_1} \right)^{\left|\frac{\lambda_2}{\lambda_1}\right|}$$

If $\left|\frac{\lambda_2}{\lambda_1}\right| = 2$

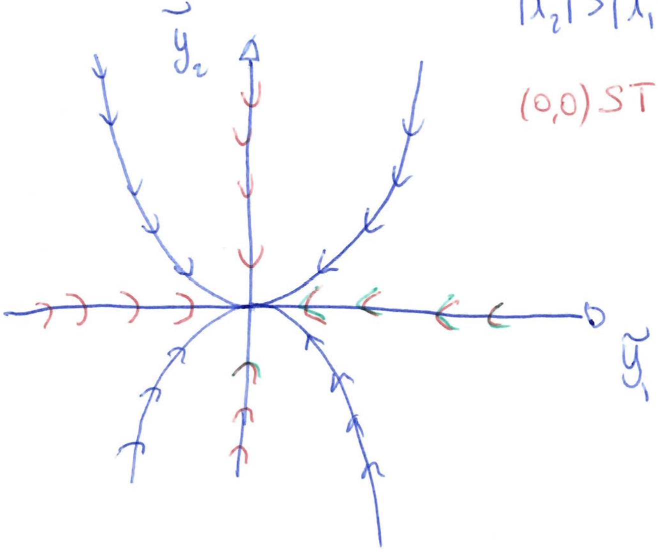
" $y = x^2$ "

These are parabolas

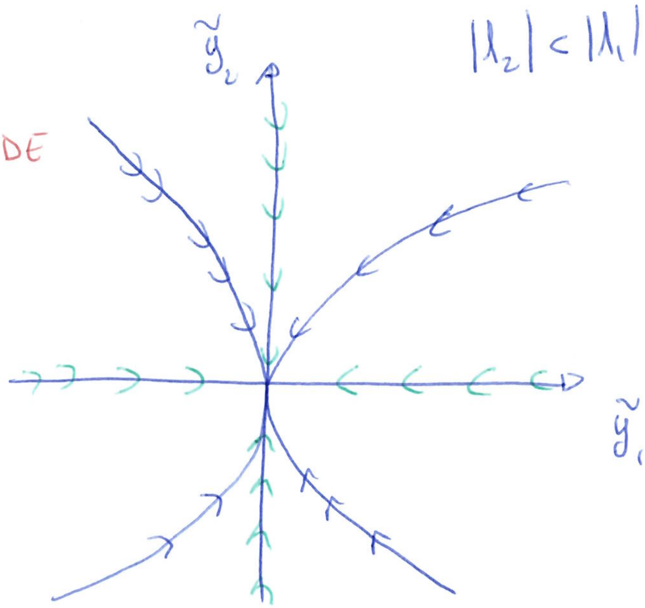
$$\begin{cases} \tilde{y}_1(t) = D_1 e^{\lambda_1 t} \quad (\lambda_1 < 0) \\ \tilde{y}_2(t) = D_2 e^{\lambda_2 t} \quad (\lambda_2 < 0) \end{cases} \qquad \text{As } t \to \infty \qquad \begin{array}{l} \tilde{y}_1 \to 0 \\ \tilde{y}_2 \to 0 \end{array}$$

$|\lambda_2| > |\lambda_1|$

(0,0) STABLE NODE



$|\lambda_2| < |\lambda_1|$



Phase portraits: distinct and complex conjugate eigenvalues

We will consider a linear system of ODEs

$$\dot{Y} = AY$$

with A a real 2×2 matrix with complex conjugate eigenvalues $\lambda_1 \neq \lambda_2$

$$\boxed{\begin{array}{l} \lambda_1 = \alpha + i\beta \\ \lambda_2 = \alpha - i\beta \end{array} \quad , \ \alpha, \beta \in \mathbb{R} \qquad \beta > 0}$$

We make the following considerations.

1) In this case the eigenvectors u_1 and u_2 are complex conjugate.

<u>Example</u> $u_1 = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}$ $\qquad u_2 = \begin{pmatrix} 1 \\ -i\omega \end{pmatrix}$

2) Therefore we can write

$$\boxed{\begin{aligned} u_1 &= v_1 - i v_2 \qquad \text{with } v_1 = \mathrm{Re}\, u_1 \\ u_2 &= v_1 + i v_2 \qquad \qquad v_2 = -\mathrm{Im}\, u_1 \end{aligned}}$$

v_1 & v_2 ~~a~~. have real components!

Example

$$u_1 = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}, \quad v_1 = \mathrm{Re}\, u_1 = \mathrm{Re}\begin{pmatrix} 1 \\ i\omega \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2 = -\mathrm{Im}\, u_1 = -\mathrm{Im}\begin{pmatrix} 1 \\ +i\omega \end{pmatrix} = -\begin{pmatrix} 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega \end{pmatrix}$$

$$u_1 = v_1 - i v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ -\omega \end{pmatrix} = \begin{pmatrix} 1 \\ i\omega \end{pmatrix}$$

$$u_2 = v_1 + i v_2$$

(3) v_1, v_2 are linearly independent

Therefore any 2-dimensional vector can be written in a unique way as a linear combination of v_1 and v_2

$$\boxed{Y = \tilde{y}_1 v_1 + \tilde{y}_2 v_2}$$

$\tilde{y}_1, \tilde{y}_2$ define a new set of coordinates.

(4) The explicit solution of $\dot{Y} = AY$ reads

$$Y = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

If y_1, y_2 are real, then D_1, D_2 are <u>complex conjugate</u>

We can put $$D_1 = \frac{1}{2}(\tilde{a} + i\tilde{b})$$

$$D_2 = \frac{1}{2}(\tilde{a} - i\tilde{b})$$

where $\tilde{a}, \tilde{b} \in \mathbb{R}$ arbitrary constants fixed by the initial conditions.