Phase portrait: Case of distinct, real eigenvalues

Consider the linear system of ODEs

$$\dot{Y} = AY \qquad \text{where} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and A is a real 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with two distinct real eigen values $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$

and two eigenvectors

$$u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \quad \text{corresponding to } \lambda_1$$

$$u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \quad \text{corresponding to } \lambda_2$$

Since the eigen vectors are <u>linearly independent</u> there is a unique way to decompose Y into a linear combination of u_1 and u_2

$$\boxed{Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2}$$

Proposition

The vector $\tilde{Y} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}$ of the new coordinates

follows the dynamical system

$$\dot{\tilde{Y}} = \tilde{A}\,\tilde{Y}$$

where $\tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Proof Since $Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2$ we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \tilde{y}_1 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} + \tilde{y}_2 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} \tilde{y}_1 p_1 + \tilde{y}_2 p_2 \\ \tilde{y}_1 q_1 + \tilde{y}_2 q_2 \end{pmatrix}$$

This transformation can be written in matrix form as

$$\boxed{Y = U\tilde{Y} \qquad \text{where } U = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}}$$

Indeed

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} p_1 \tilde{y}_1 + p_2 \tilde{y}_2 \\ q_1 \tilde{y}_1 + q_2 \tilde{y}_2 \end{pmatrix}$$

- Since u_1 and u_2 are linearly independent, $\det U \neq 0$ and therefore U is INVERTIBLE.

Therefore $\quad U^{-1} Y = \underbrace{U^{-1} U}_{Id} \tilde{Y}$ implies $\quad \tilde{Y} = U^{-1} Y \quad *$

- Differentiating with respect to time Eq. (*) we get

$$\dot{\tilde{Y}} = U^{-1} \dot{Y} \underset{\substack{\uparrow \\ \text{using} \\ \dot{Y} = AY}}{=} U^{-1} A Y \underset{\substack{\uparrow \\ \text{using} \\ Y = U\tilde{Y}}}{=} U^{-1} A U \tilde{Y}$$

Putting $\quad U^{-1} A U = \tilde{A}$ we have shown that

$$\boxed{\dot{\tilde{Y}} = \tilde{A} \tilde{Y} \qquad \text{where} \qquad \tilde{A} = U^{-1} A U}$$

U is the matrix of eigenvectors of A so it diagonalizes matrix A therefore

$$\tilde{A} = U^{-1} A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\square$

It follows that

$$\begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{pmatrix} = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} \Rightarrow \begin{cases} \dot{\tilde{y}}_1 = l_1 \tilde{y}_1 \\ \dot{\tilde{y}}_2 = l_2 \tilde{y}_2 \end{cases}$$

$\tilde{y}_1, \tilde{y}_2$ obey $\begin{cases} \tilde{y}_1 = D_1 e^{l_1 t} \\ \tilde{y}_2 = D_2 e^{l_2 t} \end{cases}$ with $Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2$

In other words $\tilde{y}_1(t) = c_1(t)$, $\tilde{y}_2(t) = c_2(t)$ with $c_1(t)$ $c_2(t)$ indicating the notation used in week 9.

What is the phase portrait in the new coordinates $\tilde{y}_1, \tilde{y}_2$?

<u>Example</u> $\dot{Y} = AY$ $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

having eigenvalues and eigenvectors

$l_1 = 2$ $u_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$l_2 = -1$ $u_2 = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

It follows that U is given by

$$U = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad \det U = 1 - 2 = -1$$

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} q_2 & -p_2 \\ -q_1 & p_1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{A} = U^{-1} A U = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} =$$

$$\underset{\substack{\uparrow \\ \text{check at} \\ \text{home}}}{=} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

We have that

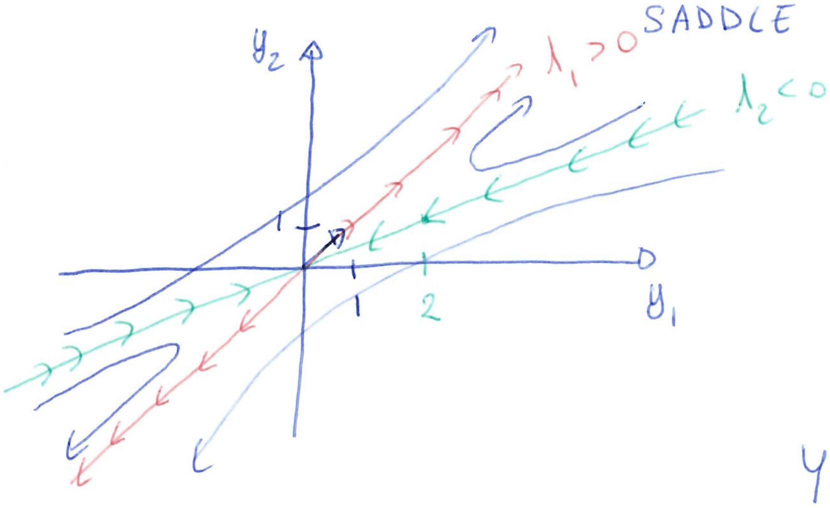
$$Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2 \qquad \text{with} \qquad \dot{\tilde{Y}} = \tilde{A} \tilde{Y}$$

Therefore $\tilde{y}_1$ and $\tilde{y}_2$ can be interpreted as new coordinates describing the trajectory on the basis of the eigenvectors u_1 and u_2

We can therefore not only consider the phase portrait in the original coordinates (y_1, y_2) but also in the new coordinates $(\tilde{y}_1, \tilde{y}_2)$

[Note that only phase portraits in the original coordinates are examinable]

<u>Original coordinates</u> (see example week 9)

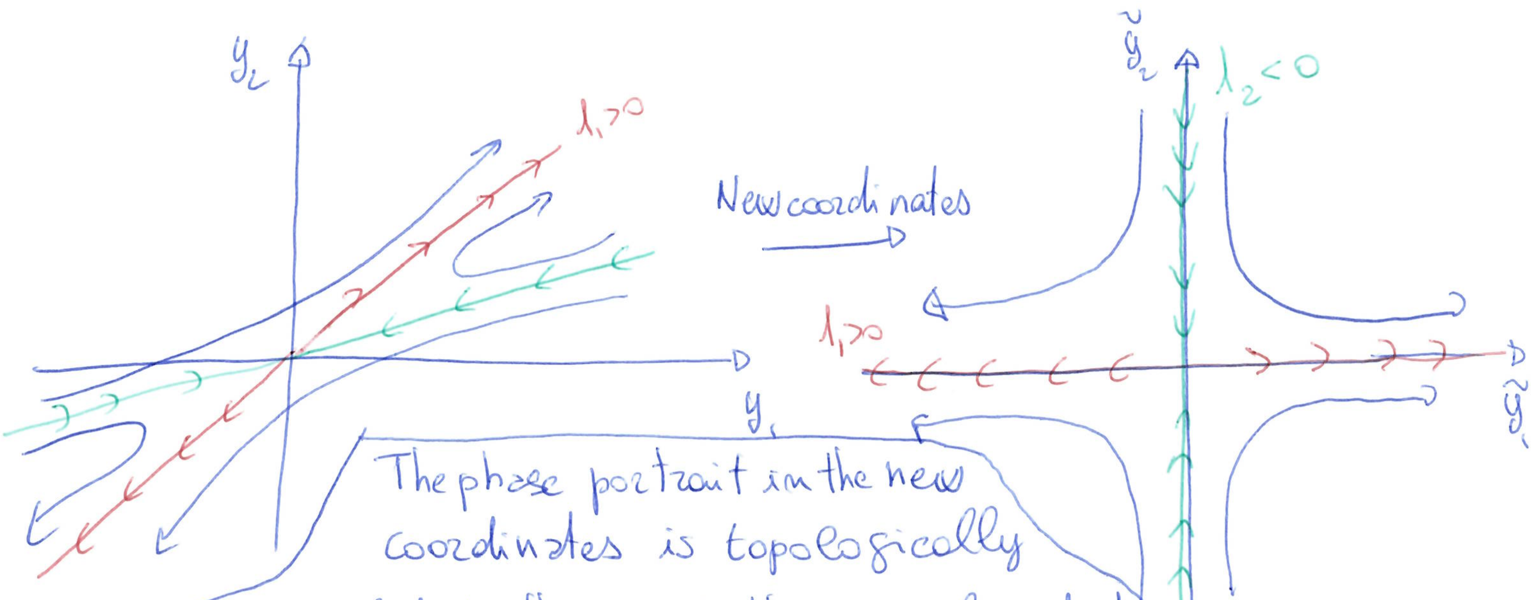


$\lambda_1 = 2 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$Y = \tilde{y}_1 u_1 + \tilde{y}_2 u_2$$

The phase portrait can be plotted in the new coordinates



The phase portrait in the new coordinates is topologically equivalent to the one in the original coordinates