Invariant Manifolds

Example Find the solution to the linear system of ODEs

$$\dot{Y} = AY$$

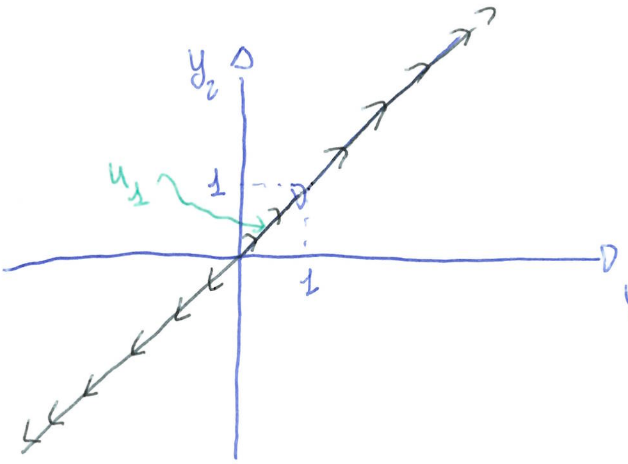
where $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$ of eigenvalues and eigenvectors

$\lambda_1 = 2 \qquad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The general solution is

$$Y(t) = D_1 e^{2t} u_1 + D_2 e^{-t} u_2$$



Any IVP with I.C. $Y(0) = k u_1$ with $k \in \mathbb{R}$ has a trajectory that moves along the direction of u_1 and moves AWAY FROM THE ORIGIN

$\boxed{\text{This is generic if } \lambda_1 \in \mathbb{R} \text{ and } \lambda_1 > 0, \ \lambda_1 \neq \lambda_2}$

$\boxed{\text{The direction defined by } u_1 \text{ is an UNSTABLE INVARIANT MANIFOLD}}$

What happens if $Y(0) = k\,u_2$ with $k \in \mathbb{R}$?

Find the solution to the IVP

$$\dot{Y} = AY \qquad \text{with I.C.} \qquad Y(0) = k u_2 \,, \quad k \in \mathbb{R}$$

The general solution

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

Imposing the I.C.

$$Y(0) = D_1 u_1 + D_2 u_2 = k u_2$$

Rearranging

$$\underbrace{(D_1)}_{=0} u_1 + \underbrace{(D_2 - k)}_{=0} u_2 = 0$$

$$\begin{cases} D_1 = 0 \\ D_2 = k \end{cases}$$

Solution to the IVP

$$Y(t) = k e^{\lambda_2 t} u_2 = k e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

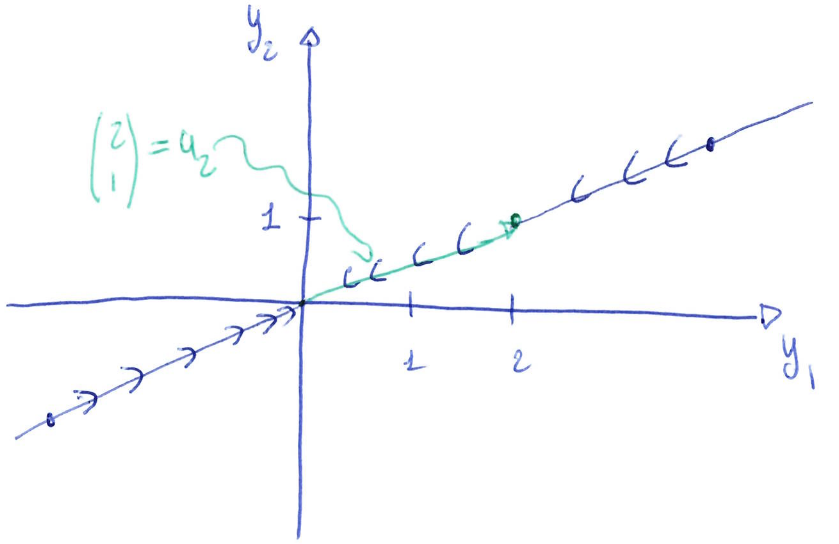
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = k e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solution to the IVP.

$$\begin{cases} y_1 = 2k e^{-t} \\ y_2 = k e^{-t} \end{cases}$$

$$\boxed{y_1 = 2 y_2}$$

Trajectory



Trajectory

$y_2 = \frac{1}{2} y_1$

The direction determined by u_2 is an STABLE INVARIANT MANIFOLD

If $k>0$ $t \to \infty$ $y_1 \to 0^+$ $y_2 \to 0^+$

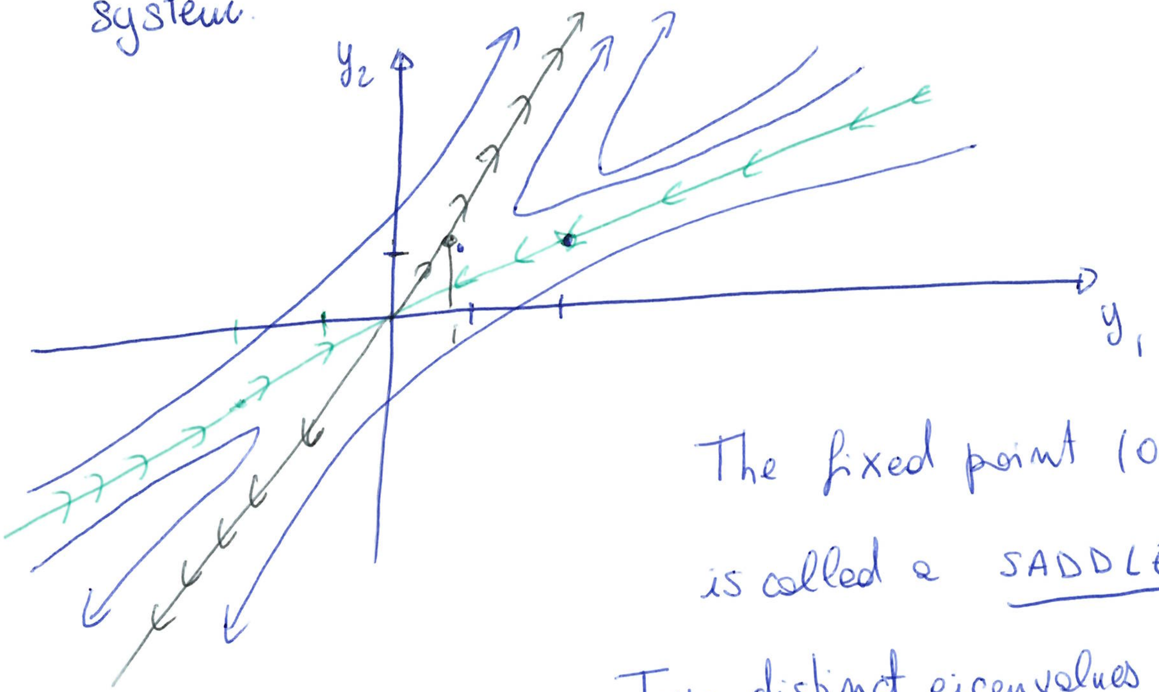
If $k<0$ $t \to \infty$ $y_1 \to 0^-$ $y_2 \to 0^-$

All trajectories move along the direction of u_2 and move TOWARD THE ORIGIN.

$\boxed{\text{This is generic if } \lambda_2 \in \mathbb{R},\ \lambda_2 < 0 \quad \lambda_1 \neq \lambda_2}$

Stable invariant manifols are identified by the direction of eigenvector associated to real and negative eigenvalues.

Let us put together the phase portrait of our dynamical system.



The fixed point $(0,0)$ is called a <u>SADDLE</u>

Two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ of opposite sign.

<u>Introduction to phase portraits</u>

Consider the system of ODES

$$\dot{Y} = AY$$

with A indicating a 2×2 matrix having two distinct eigenvalues $\lambda_1 \neq \lambda_2$

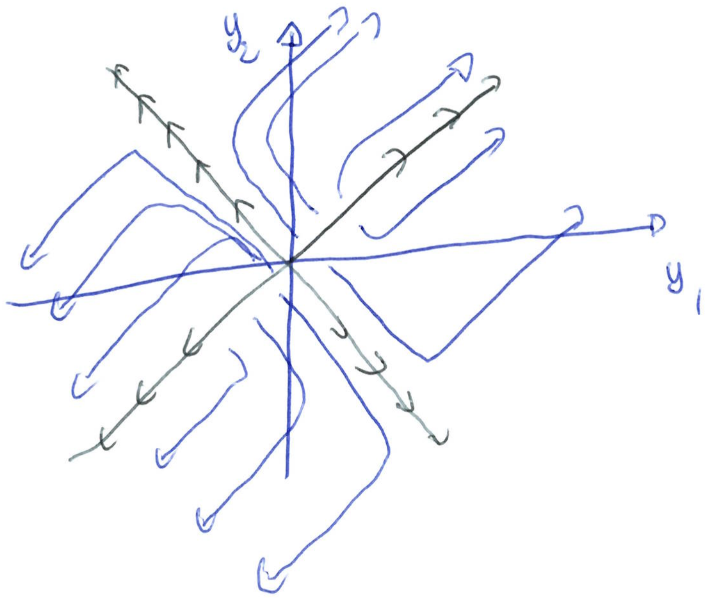
What can we say about phase portraits?

Knowing the eigenvalues we can already draw important conclusions but for details we will need information about eigenvectors.

① $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

a) $\lambda_1 > 0 \quad \lambda_2 > 0$

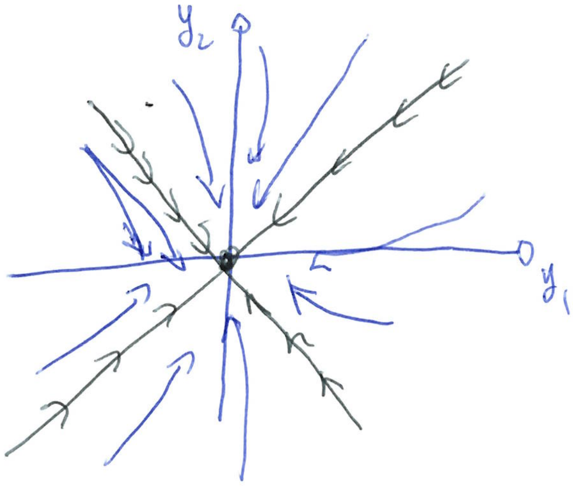
u_1, u_2 will both define UNSTABLE INVARIANT MANIFOLDS.



$(0,0)$ is an UNSTABLE NODE

b) $\lambda_1 < 0 \quad \lambda_2 < 0$

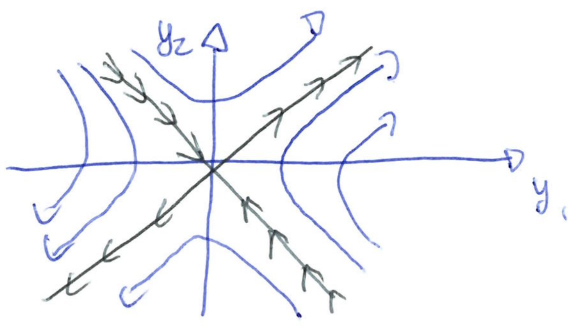
u_1, u_2 will both define STABLE INVARIANT MANIFOLDS



$(0,0)$ is a STABLE NODE

c) $\lambda_1 > 0 \quad \lambda_2 < 0 \rightarrow$
(σ $\lambda_1 < 0 \quad \lambda_2 > 0$)

$u_1 \rightarrow$ UNSTABLE MANIFOLD
$u_2 \rightarrow$ STABLE MANIFOLD

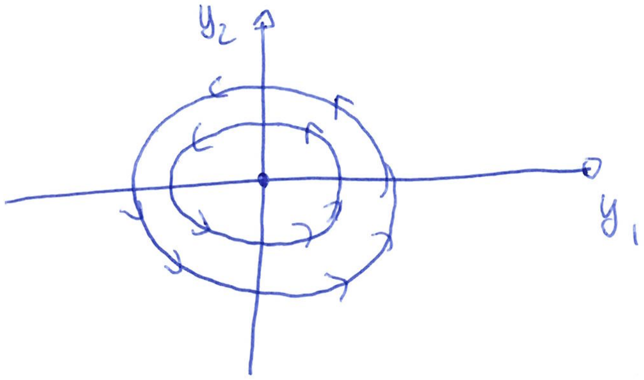


$(0,0)$ is a SADDLE

(B) $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \in \mathbb{R}$ $\beta \neq 0$

Complex conjugate eigenvalue.

(a) If $\alpha = 0$ (0,0) is a CENTRE



All trajectories are periodic

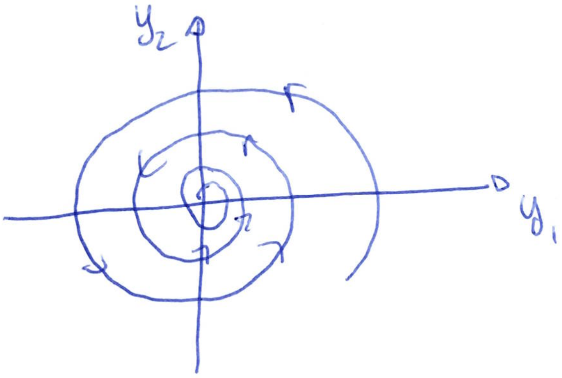
Periodic orbits

To check the direction (clockwise or anti clockwise) you need to conside more details

(0,0) is a STABLE FOCUS

(b) If $Re\,\alpha < 0$

The phase portrait is a SPIRAL IN

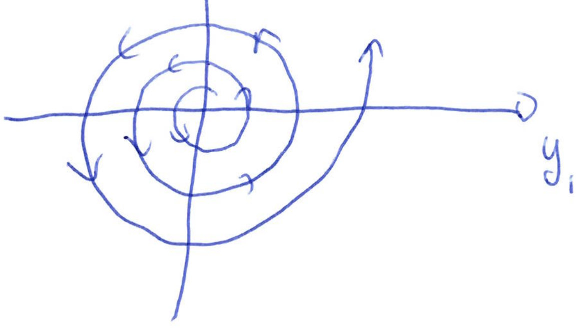


All trajectories will spiral into the origin.

(0,0) is a UNSTABLE FOCUS

(c) If $Re\,\alpha > 0$

The phase portrait is a SPIRAL OUT



All trajectories spiral out from the origin.

Already knowing the eigenvalues of A we can draw important conclusions about the phase portraits of a linear system.